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The generalized von Neumann-Jordan type constant and fixed points for multivalued nonexpansive mappings

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ABSTRACT: Some sufficient conditions for the Domínguez-Lorenzo condition in terms of the generalized von Neumann-Jordan type constant $C_{-\infty}^{(p)}(X)$, the coefficient of weak orthogonality $\mu(X)$, the Domínguez Benavides coefficient $R(1,X)$ were given in this paper, which imply the existence of fixed point for multivalued nonexpansive mappings. Moreover, these results improve some well known conclusion in the recent literature.

KEYWORDS: generalized von Neumann-Jordan type constant, coefficient of weak orthogonality, Domínguez Benavides coefficient, multivalued nonexpansive mapping, fixed point

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INTRODUCTION

In 1969, Nadler[1] established the multivalued version of Banach contraction principle. Since then, the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multivalued nonexpansive mappings. However, the fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings, many questions remain open, for instance, the possibility of extending the well-known Kirk's theorem[2], that is, do Banach spaces with normal structure have the fixed point property (FPP) for multivalued nonexpansive mappings? Since normal structure is implied by different geometric properties of Banach spaces, it is natural to study if those properties imply the FPP for multivalued mappings. Dhompongsa et al[3] introduced the Domínguez-Lorenzo condition ((DL)-condition) which implies the FPP for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply the (DL)-condition. In 2007, Benavides and Gavira[4] had established FPP for multivalued nonexpansive mappings in terms of the modulus of squareness, universal infinite-dimensional modulus, and Opial modulus. Kaewkhao[5] had established FPP for multivalued nonexpansive mappings in terms of the James constant, the von Neumann-Jordan constant, the coefficient of weak orthogonality. In 2010, Benavides and Gavira[6] had given a survey of this subject and presented the main known results and current research directions. In particular, the von Neumann-Jordan constant, Zbăganu constant, and von Neumann-Jordan type constant play an important role in the description of various geometric structures. Therefore, some recent studies have focused on these constants. A Banach space X with any of the following conditions satisfies the (DL)-condition[5, 7–10]:

(i) $C_{\mathrm{NJ}}(X) < 1 + 1/\mu(X)^2$;
(ii) $C_{\mathrm{NJ}}(X) < [1 + 1/R(1,X)]^2/2$;
(iii) $C_{\mathrm{Z}}(X) < [1 + 1/R(1,X)]^2/2$;
(iv) $C_{\mathrm{Z}}(X) < 1 + 1/\mu(X)^2$;
(v) $C_{-\infty}(X) < 1 + 1/\mu(X)^2$.

PRELIMINARIES

Throughout this paper, S_X and B_X denote the unit sphere and the unit ball of X, respectively. Before going to the results, let us recall some concepts and results which will be used in the following sections. The following constants of a Banach space X,

$$J(X)=\sup\{\min\{\|x+y\|,\|x-y\|\}:x,y\in S_X\},$$

$$C_{\mathrm{NJ}}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}:\right.\\ \left. x,y\in X,(x,y)\neq(0,0)\right\},$$

$$C_{\mathrm{Z}}(X)=\sup\left\{\frac{\|x+y\|\|x-z\|}{\|x\|^2+\|y\|^2}:\right.\\ \left. x,y\in X,(x,y)\neq(0,0)\right\},$$

$$C_{-\infty}(X)=\sup\left\{\frac{\min\{\|x+y\|^2,\|x-y\|^2\}}{\|x\|^2+\|y\|^2}:\right.\\ \left. x,y\in X,(x,y)\neq(0,0)\right\}$$

are called the James constant, von Neumann-Jordan constant, Zbăganu constant, and von-Jordan Neumann type constant, respectively. These constants have been considered in many papers, some properties and inequalities among them have been indicated[3,5–15],

$$C_{-\infty}(X)\leqslant C_{\mathrm{Z}}(X)\leqslant C_{\mathrm{NJ}}(X)\leqslant J(X).$$

Moreover, the above inequalities are strict in some Banach spaces[9–11,14,15].

Recently, von Neumann-Jordan constant, Zbăganu constant, and von Neumann-Jordan type constant are generalized in the following ways: for $1\leqslant p<\infty$,

$$C^{(p)}_{\mathrm{NJ}}(X)=\sup\left\{\frac{\|x+y\|^p+\|x-y\|^p}{2^{p-1}(\|x\|^p+\|y\|^p)}:\right.\\ \left. x,y\in X,(x,y)\neq(0,0)\right\},$$

$$C^{(p)}_{\mathrm{Z}}(X)=\sup\left\{\frac{\|x+y\|^{\frac{p}{2}}\|x-y\|^{\frac{p}{2}}}{2^{p-2}(\|x\|^p+\|y\|^p)}:\right.\\ \left. x,y\in X,(x,y)\neq(0,0)\right\},$$

$$C^{(p)}_{-\infty}(X)=\sup\left\{\frac{\min\{\|x+y\|^p,\|x-y\|^p\}}{2^{p-2}(\|x\|^p+\|y\|^p)}:\right.\\ \left. x,y\in X,(x,y)\neq(0,0)\right\}.$$

From the definitions of generalized von Neumann-Jordan constant, generalized Zbăganu constant, and generalized von Neumann-Jordan type constant, it is obvious that $C^{(2)}_{\mathrm{NJ}}(X)=C_{\mathrm{NJ}}(X)$, $C^{(2)}_{\mathrm{Z}}(X)=C_{\mathrm{Z}}(X)$, and $C^{(2)}_{-\infty}(X)=C_{-\infty}(X)$.

The coefficient $R(1,X)$ is introduced by Benavide[16],

$$R(1,X)=\sup\big\{\liminf_{n\to\infty}\|x_n+x\|\big\},$$

where the supremum is taken over all $x\in X$ with $\|x\|\leqslant 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D[(x_n)]:=\limsup_{n\to\infty}\ \limsup_{m\to\infty}\|x_n-x_m\|\leqslant 1.$$

It is clear that $1\leqslant R(1,X)\leqslant 2$. Some sufficient conditions for the (DL)-condition in terms of this coefficient have been studied[13,15].

Some geometric properties of Banach spaces in terms of the above constants are investigated in Refs. 11, 14.

(1) $2^{2-p}\leqslant C^{(p)}_{-\infty}(X)\leqslant C^{(p)}_{\mathrm{Z}}(X)\leqslant C^{(p)}_{\mathrm{NJ}}(X)\leqslant 2$ for all $1\leqslant p<\infty$. Moreover, the inequalities are strict in some Banach spaces.

(2) Let $1\leqslant p<\infty$, then $R(1,X)\leqslant J(X)\leqslant 2^{\frac{p-1}{p}}\sqrt[p]{C^{(p)}_{-\infty}(X)}$.

(3) X is uniformly nonsquare if and only if $C^{(p)}_{-\infty}(X)<2$ for some $1\leqslant p<\infty$.

Jiménez-Melado and Llorens-Fuster defined the coefficient of weak orthogonality[12]:

$$\mu(X)=\inf\{\lambda:\limsup_{n\to\infty}\|x_n+x\|\leqslant\lambda\limsup_{n\to\infty}\|x_n-x\|\},$$

where the infimum is taken over all $x\in X$ and all weakly null sequence $\{x_n\}$. It is well known that $1\leqslant\mu(X)\leqslant 3$, furthermore it is proved that $\mu(X)=\mu(X^*)$ in reflexive Banach space.

In the sequent, some concepts and results of multivalued mapping were introduced. Let C be a nonempty subset of a Banach space X, we shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X, and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T:C\to CB(X)$ is said to be nonexpansive if

$$H(Tx,Ty)\leqslant\|x-y\|,\quad\forall\,x,y\in C,$$

where $H(.,.)$ denotes the Hausdorff metric on $CB(X)$ defined by, for $A,B\in CB(X)$,

$$H(A,B):=\max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\sup_{y\in B}\inf_{x\in A}\|x-y\|\}.$$

Let $\{x_n\}$ be a bounded sequence in X, the asymptotic radius $r(C,\{x_n\})$ and the asymptotic center $A(C\{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C,\{x_n\}) = \inf\{\limsup_n \|x_n - x\| : x \in C\},$$
$$A(C,\{x_n\}) = \{x \in C : \limsup_n \|x_n - x\| = r(C,\{x_n\})\},$$

respectively. It is well known that $A(C,\{x_n\})$ is a nonempty weakly compact convex set whenever C is. The sequence $\{x_n\}$ is called regular with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x\in C} \sup_{y\in D} \|x - y\|.$$

The (DL)-condition is defined as follows:

Definition 1 If there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C, which is regular with respect to C,

$$r_C(A(C,\{x_n\})) \leqslant \lambda\, r(C,\{x_n\}).$$

The following results show that the (DL)-condition is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive mappings[3].

Theorem 1 *Let X be a Banach space satisfying the (DL)-condition, then X has weak normal structure.*

Theorem 2 *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies the (DL)-condition, and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

MAIN RESULTS

Theorem 3 *Let C be a weakly compact convex subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in C, regular with respect to C, then*

$$r_C(A(C,\{x_n\})) \leqslant \frac{\mu(X)\sqrt[p]{2^{p-2}C_{-\infty}^{(p)}(X)(\mu(X)^p+1)}}{\mu(X)^2+1}\, r(C,\{x_n\}).$$

Proof: Denote $r = r(C,\{x_n\})$ and $A = A(C,\{x_n\})$. By passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Let $z \in A$, then $\limsup_n \|x_n - z\| = r$. Denote $\mu = \mu(X)$. The definition of $\mu(X)$ gives

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \|(x_n - x) + (z - x)\| \\ &\leqslant \mu \limsup_n \|(x_n - x) - (z - x)\| = \mu r.\end{aligned}$$

On the other hand, by the weak lower semicontinuity of the norm,

$$\liminf_n \|(\mu^2 - 1)(x_n - x) - (\mu^2 + 1)(z - x)\| \geqslant (\mu^2 + 1)\|z - x\|.$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that
(1) $\|x_N - z\| \leqslant r + \varepsilon$,
(2) $\|x_N - 2x + z\| \leqslant \mu(r + \varepsilon)$,
(3) $\left\|x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geqslant r - \varepsilon$, and
(4) $\left\|(\mu^2 - 1)(x_N - x) - (\mu^2 + 1)(z - x)\right\| \geqslant (\mu^2 + 1)\|z - x\|(\frac{r-\varepsilon}{r})$.

Put $u = \mu^2(x_N - z)$ and $v = (x_N - 2x + z)$, the above estimates give $\|u\| \leqslant \mu^2(r + \varepsilon)$, $\|v\| \leqslant \mu(r + \varepsilon)$, so that

$$\begin{aligned}\|u + v\| &= \left\|\mu^2((x_N - x) - (z - x)) + (x_N - x) + (z - x)\right\| \\ &= (\mu^2 + 1)\left\|(x_N - x) - \frac{\mu^2 - 1}{\mu^2 + 1}(z - x)\right\| \\ &= (\mu^2 + 1)\left\|x_N - \left(\frac{2}{\mu^2 + 1}x + \frac{\mu^2 - 1}{\mu^2 + 1}z\right)\right\| \\ &\geqslant (\mu^2 + 1)(r - \varepsilon),\end{aligned}$$

$$\begin{aligned}\|u - v\| &= \left\|\mu^2((x_N - x) - (z - x)) - (x_N - x) - (z - x)\right\| \\ &= \left\|(\mu^2 - 1)(x_N - x) - (\mu^2 + 1)(z - x)\right\| \\ &\geqslant (\mu^2 + 1)\|z - x\|\Big(\frac{r - \varepsilon}{r}\Big).\end{aligned}$$

By the definition of $C_{-\infty}^{(p)}(X)$ and since $\|z - x\| \leqslant r$,

$$\begin{aligned}C_{-\infty}^{(p)}(X) &\geqslant \frac{\min\{\|u + v\|^p, \|u - v\|^p\}}{2^{p-2}(\|u\|^p + \|v\|^p)} \\ &\geqslant \frac{\min\{(\mu^2 + 1)^p(r - \varepsilon)^p, (\mu^2 + 1)^p\|z - x\|^p\left(\frac{r-\varepsilon}{r}\right)^p\}}{2^{p-2}\mu^p(\mu^p + 1)(r + \varepsilon)^p}.\end{aligned}$$

Let $\varepsilon \to 0^+$, we obtain that

$$C^{(p)}_{-\infty}(X) \geqslant \frac{(\mu^2+1)^p \|z-x\|^p}{2^{p-2}\mu^p(\mu^p+1)r^p},$$

then

$$\|z-x\| \leqslant \frac{\mu \sqrt[p]{2^{p-2}C_{-\infty}(X)(\mu^p+1)}}{\mu^2+1}\, r.$$

This holds for arbitrary $z \in A$, hence

$$r_C(A) \leqslant \frac{\mu \sqrt[p]{2^{p-2}C^{(p)}_{-\infty}(X)(\mu^p+1)}}{\mu^2+1}\, r.$$

□

Corollary 1 *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C^{(p)}_{-\infty}(X) < (\mu(X)^2+1)^p/2^{p-2}\mu(X)^p(\mu(X)^p+1)$, and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Proof: From the condition

$$C^{(p)}_{-\infty}(X) < \frac{(\mu(X)^2+1)^p}{2^{p-2}\mu(X)^p(\mu(X)^p+1)}$$

then X satisfies the (DL)-condition by Theorem 3, therefore T has a fixed point by Theorem 2. □

Corollary 2 *Let X be a Banach space such that $C^{(p)}_{-\infty}(X) < (\mu(X)^2+1)^p/2^{p-2}\mu(X)^p(\mu(X)^p+1)$, then X has normal structure.*

Proof: Firstly, from Theorem 1 and Theorem 3, it is easy to prove that X has weak normal structure. Secondly, we have

$$C^{(p)}_{-\infty}(X) < \frac{(\mu(X)^2+1)^p}{2^{p-2}\mu(X)^p(\mu(X)^p+1)} < 2$$

for which $1 \leqslant \mu(X) \leqslant 3$. This implies that X is uniformly nonsquare, then X is reflexive, therefore weak normal structure coincides with normal structure. □

In particular, letting $p=2$ in Corollary 1 and Corollary 2, we obtain the following corollaries[10].

Corollary 3 *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{-\infty}(X) < 1+1/\mu(X)^2$, and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Corollary 4 *If $C_{-\infty}(X) < 1+1/\mu(X)^2$, then X has normal structure.*

Theorem 4 *Let C be a weakly compact convex subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in C, regular with respect to C, then*

$$r_C(A(C,\{x_n\})) \leqslant \frac{R(1,X)\sqrt[p]{2^{p-1}C^{(p)}_{-\infty}(X)}}{R(1,X)+1}\, r(C,\{x_n\}).$$

Proof: Denote $r = r(C,\{x_n\})$ and $A = A(C,\{x_n\})$. We assume $r > 0$. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Consequently, we assume $\{x_n\}$ is weakly convergent to a point $x \in C$ and $d = \lim_{n\neq m}\|x_n - x_m\|$ exists. Observe that the norm is weak lower semicontinuous, then

$$\begin{aligned}\liminf_n \|x_n - x\| &\leqslant \liminf_n \liminf_m \|x_n - x_m\| \\ &= \lim_{n\neq m}\|x_n - x_m\| = d.\end{aligned}$$

Let $\epsilon > 0$, taking a subsequence if necessary, we can assume that $\|x_n - x\| < d+\epsilon$ for all n. Let $z \in A$, then $\limsup_n \|x_n - z\| = r$ and $\|x-z\| \leqslant \liminf_n \|x_n - z\| \leqslant r$. Denote $R = R(1,X)$, by the definition of $R(1,X)$, we have

$$\begin{aligned}\liminf_n &\left\|\frac{x_n - x}{d+\epsilon} + \frac{z-x}{r}\right\| \\ &= \liminf_n \left\|\frac{x_n - x}{d+\epsilon} - \frac{x-z}{r}\right\| \leqslant R.\end{aligned}$$

Convexity of C implies that $\frac{R-1}{R+1}x + \frac{2}{R+1}z \in C$. Since the norm is weak lower semicontinuous, then

$$\begin{aligned}\liminf_n &\left\|\frac{x_n - z}{r} + \frac{1}{R}\left(\frac{x_n - x}{d+\epsilon} - \frac{x-z}{r}\right)\right\| \\ &= \liminf_n \left\|\left(\frac{1}{r} + \frac{1}{R(d+\epsilon)}\right)(x_n - x) \right. \\ &\quad \left. + \left(\frac{1}{r} - \frac{1}{Rr}\right)x - \left(\frac{1}{r} - \frac{1}{Rr}\right)z\right\| \\ &\geqslant \left\|\left(\frac{1}{r} - \frac{1}{Rr}\right)x + \frac{2}{Rr}z - \left(\frac{1}{r} + \frac{1}{Rr}\right)z\right\| \\ &= \left(\frac{1}{r} + \frac{1}{Rr}\right)\left\|\frac{R-1}{R+1}x + \frac{2}{R+1}z - z\right\| \\ &\geqslant \left(1 + \frac{1}{R}\right)\left(\frac{r_C(A)}{r}\right),\end{aligned}$$

$$\begin{aligned}\liminf_n \frac{1}{Rr} &\left\|R(x_n - z) - \left(\frac{r(x_n - x)}{d+\epsilon} - (x-z)\right)\right\| \\ &\geqslant \frac{1}{Rr}\left\|\left(R - \frac{r}{d+\epsilon}\right)(x_n - x) + (R+1)(x-z)\right\| \\ &\geqslant \left(1 + \frac{1}{R}\right)\left(\frac{r_C(A)}{r}\right).\end{aligned}$$

For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
(1) $\|x_N - z\| \leqslant r + \varepsilon$,
(2) $\left\| \dfrac{r(x_N - x)}{d+\epsilon} - (x - z) \right\| \leqslant R(r+\epsilon)$,
(3) $\left\| R(x_N - z) + \dfrac{r(x_N - x)}{d+\epsilon} - (x - z) \right\| \geqslant (R+1) r_C(A) \Big(\dfrac{r-\epsilon}{r}\Big)$,
(4) $\left\| R(x_N - z) - \Big(\dfrac{r(x_N - x)}{d+\epsilon} - (x - z)\Big) \right\| \geqslant (R+1) r_C(A) \Big(\dfrac{r-\epsilon}{r}\Big)$.

Now, put $u = R(x_N - z)$ and $v = \big(\frac{r(x_N - x)}{d+\epsilon} - (x - z)\big)$. Using the above estimates, we obtain $\|u\| \leqslant R(r+\epsilon)$, $\|v\| \leqslant R(r+\epsilon)$,

$$\begin{aligned} \|u+v\| &= \left\| R(x_N - z) + \frac{r(x_N - x)}{d+\epsilon} - (x-z) \right\| \\ &\geqslant (R+1) r_C(A) \Big(\frac{r-\epsilon}{r}\Big), \\ \|u-v\| &= \left\| R(x_n - z) - (1-a)\Big(\frac{r(x_n - x)}{d+\epsilon} - x + z\Big) \right\| \\ &\geqslant (R+1) r_C(A) \Big(\frac{r-\epsilon}{r}\Big). \end{aligned}$$

From the definition of $C_{-\infty}^{(p)}(X)$, then

$$\begin{aligned} C_{-\infty}^{(p)}(X) &\geqslant \frac{\min\{\|u+v\|^p, \|u-v\|^p\}}{2^{p-2}(\|u\|^p + \|v\|^p)} \\ &\geqslant \frac{1}{2^{p-1}} \Big(\frac{R+1}{R}\Big)^p \Big(\frac{r_C(A)}{r}\Big)^p \Big(\frac{r-\epsilon}{r+\epsilon}\Big)^p. \end{aligned}$$

Since the inequality is true for every $\epsilon > 0$, then

$$r_C(A) \leqslant \frac{R\sqrt[p]{2^{p-1} C_{-\infty}^{(p)}(X)}}{R+1} r.$$

□

Corollary 5 *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{-\infty}^{(p)}(X) < \frac{1}{2^{p-1}}\Big(\frac{R(1,X)+1}{R(1,X)}\Big)^p$, and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Proof: If $C_{-\infty}^{(p)}(X) < \frac{1}{2^{p-1}}\Big(\frac{R(1,X)+1}{R(1,X)}\Big)^p$, then X satisfies the (DL)-condition by Theorem 4, so T has a fixed point by Theorem 2. □

Corollary 6 *Let X be a Banach space such that $C_{-\infty}^{(p)}(X) < \frac{1}{2^{p-1}}\Big(\frac{R(1,X)+1}{R(1,X)}\Big)^p$, then X has normal structure.*

Proof: In fact, from Theorem 1 and Theorem 4, it is easy to prove that X has weak normal structure. Since $1 \leqslant R(1,X) \leqslant 2$, we have

$$C_{-\infty}^{(p)}(X) < \frac{1}{2^{p-1}}\Big(\frac{R(1,X)+1}{R(1,X)}\Big)^p < 2$$

for some $1 \leqslant p < \infty$. This implies that X is uniformly nonsquare, then X is reflexive, therefore weak normal structure coincides with normal structure. □

In particular, letting $p = 2$ in Corollary 5 and Corollary 6, we obtain the following corollaries.

Corollary 7 *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{-\infty}(X) < \frac{1}{2}\big(1 + \frac{1}{R(1,X)}\big)^2$, and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Corollary 8 *If $C_{-\infty}(X) < \frac{1}{2}\big(1 + \frac{1}{R(1,X)}\big)^2$, then X has normal structure.*

Remark 1 It is shown in that if $C_Z(X) < \frac{1}{2}\big(1 + \frac{1}{R(1,X)}\big)^2$, then X satisfies the (DL)-condition[8]. From the inequality $C_{-\infty}(X) \leqslant C_Z(X) \leqslant C_{NJ}(X)$, Corollary 7 is better than Zhang's result[8] (Corollary 2.2), which also improves the result of Zhang and Cui, that if $C_{\mathrm{NJ}}(X) < \frac{1}{2}\big(1 + \frac{1}{R(1,X)}\big)^2$, then X satisfies the (DL)-condition[7].

Remark 2 In the sequent, an example is given to show that some of our results are sharp. The Bynum space $l_{2,\infty}$, which is the space l_2 re-normed according to $\|x\|_{2,\infty} = \max\{\|x^+\|_2, \|x^-\|_2\}$, where x^+ and x^- are the positive and the negative part of x, respectively, defined as $x^+(i) = \max\{x(i), 0\}$ and $x^- = x^+ - x$. It is well known that $C_{-\infty}(l_{2,\infty}) = \frac{3}{2}$ and $\mu(l_{2,\infty}) = \sqrt{2}$, then

$$C_{-\infty}(l_{2,\infty}) = 1 + \frac{1}{\mu(l_{2,\infty})^2}.$$

However, $l_{2,\infty}$ lacks normal structure, therefore Bynum space $l_{2,\infty}$ does not satisfy the (DL)-condition, then the results obtained in this paper are sharp.

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