RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2019.45.275

ScienceAsia 45 (2019): **275–278**

Hyers-Ulam stability of an alternative functional equation of Jensen type

Choodech Srisawat

Department of Mathematics, Faculty of Science, Udon Thani Rajabhat University, Udon Thani 41000 Thailand

e-mail: ch.srisawat@gmail.com

Received 6 Jul 2018
Accepted 30 Jun 2019

ABSTRACT: Given an integer $\lambda \notin \{-2,-1,0,2\}$, we investigate the Hyers-Ulam stability of an alternative Jensen's functional equation $f(xy^{-1})-2f(x)+f(xy)=0$ or $f(xy^{-1})-\lambda f(x)+f(xy)=0$ where f is a mapping from an abelian group to a Banach space.

KEYWORDS: stability, alternative equation, Jensen's functional equation

MSC2010: 39B82 39B72

INTRODUCTION

The problem of the alternative Cauchy functional equation has been widely studied. For instance, Kannappan et al[1] studied the solutions of the alternative Cauchy functional equation of the form

$$\big(f(x+y)-af(x)-bf(y)\big)\big(f(x+y)-f(x)-f(y)\big)=0, \quad (1)$$

where f is a function from an abelian group to a commutative integral domain of characteristic zero. Ger[2] extended their results to the alternative functional equation

$$\big(f(x+y)-af(x)-bf(y)\big)\big(f(x+y)-cf(x)-\mathrm{d}f(y)\big)=0.$$

Forti[3] found the general solution of the alternative Cauchy functional equation of the form

$$\big(cf(x+y)-af(x)-bf(y)-d\big)\big(f(x+y)-f(x)-f(y)\big)=0.$$

Nakmahachalasint[4] first solved an alternative Jensen's functional equation of the form

$$f(x)\pm 2f(xy)+f(xy^2)=0 \quad (2)$$

on a semigroup. His research extended the work of Ng[5] and Parnami et al[6] on the classical Jensen's functional equation

$$f(xy^{-1})-2f(x)+f(xy)=0 \quad (3)$$

on a group. Nakmahachalasint[7] then investigated the Hyers-Ulam stability of the alternative Jensen's functional equation (2) in the class of mappings from 2-divisible abelian groups to Banach spaces.

Given an integer $\lambda \neq 2$, Srisawat et al[8] solved the alternative Jensen's functional equation of the form

$$\begin{aligned} &f(xy^{-1})-2f(x)+f(xy)=0 \quad \text{or} \\ &f(xy^{-1})-\lambda f(x)+f(xy)=0 \end{aligned} \quad (4)$$

when f is a function from a group to a uniquely divisible abelian group, but a stability problem has not yet been investigated. In this paper, we will prove the Hyers-Ulam stability of the alternative Jensen's functional equation (4) when $\lambda \notin \{-2,-1,0,2\}$ is an integer and f is a mapping from an abelian group $(G,\cdot)$ to a Banach space $(E,\|\cdot\|)$. In other words, for every $\varepsilon \geqslant 0$, we prove that there exist $\delta_1,\delta_2 \geqslant 0$ such that if, for an integer $\lambda \notin \{-2,-1,0,2\}$, a mapping $f: G \to E$ satisfies the inequalities

$$\begin{aligned} &\|f(xy^{-1})-2f(x)+f(xy)\| \leqslant \delta_1 \quad \text{or} \\ &\|f(xy^{-1})-\lambda f(x)+f(xy)\| \leqslant \delta_2 \end{aligned} \quad (5)$$

for every $x,y \in G$, then there exists a unique Jensen's mapping $J: G \to E$ with $\|f(x)-J(x)\| \leqslant \varepsilon$ for all $x \in G$.

AUXILIARY LEMMAS

Throughout this study, we let $(G,\cdot)$ be a group and $(E,\|\cdot\|)$ be a Banach space. Given an integer λ, and a function $f: G \to E$. For $x, y \in G$, we define

$$\mathscr{F}_y^{(\lambda)}(x) := \|f(xy^{-1}) - \lambda f(x) + f(xy)\|.$$

Furthermore, for $\delta_1, \delta_2 \geqslant 0$ and $\lambda \notin \{-2,-1,0,2\}$, we let

$$\mathscr{S} f_y^{(\lambda)}(x) := \big(\mathscr{F}_y^{(2)}(x) \leqslant \delta_1 \quad \text{or} \quad \mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2\big),$$
$$\delta := \max\{\delta_1, \delta_2\}, \quad \text{and}$$

$$\mathscr{M}_\delta^\lambda := \big(29 + 42|\lambda| + 38\lambda^2 + 20|\lambda^3| + 4\lambda^4\big)\delta.$$

The set of all solution of (5) is denoted by

$$\mathscr{A}_{(G,E)}^{(\lambda)} := \Big\{ f: G \to E \,\Big|\, \mathscr{S} f_y^{(\lambda)}(x),\ \forall x, y \in G \Big\}.$$

We first prove the bound of $f(x)$ concerning the relation between $\mathscr{S} f_y^{(\lambda)}(xy^{-1})$, $\mathscr{S} f_y^{(\lambda)}(x)$, and $\mathscr{S} f_y^{(\lambda)}(xy)$.

Lemma 1 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$.*
(i) *If $\mathscr{F}_y^{(2)}(xy^{-1}) \leqslant \delta_1$, $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$, and $\mathscr{F}_y^{(2)}(xy) \leqslant \delta_1$, then $\|f(x)\| \leqslant 5\delta$.*
(ii) *If $\mathscr{F}_y^{(\lambda)}(xy^{-1}) \leqslant \delta_2$, $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$, and $\mathscr{F}_y^{(\lambda)}(xy) \leqslant \delta_2$, then $\|f(x)\| \leqslant (3 + |\lambda|)\delta$.*

Proof:
(i) We observe that

$$\begin{aligned} &\|f(xy^{-2}) + 2(1-\lambda)f(x) + f(xy^2)\| \\ &\leqslant \mathscr{F}_y^{(2)}(xy^{-1}) + 2\mathscr{F}_y^{(\lambda)}(x) + \mathscr{F}_y^{(2)}(xy) \\ &\leqslant 4\delta. \end{aligned} \tag{6}$$

Consider the alternatives in $\mathscr{S} f_{y^2}^{(\lambda)}(x)$. The inequality $\mathscr{F}_{y^2}^{(2)}(x) \leqslant \delta_1$ and (6) give $\|(4 - 2\lambda)f(x)\| \leqslant 5\delta$, while the inequality $\mathscr{F}_{y^2}^{(\lambda)}(x) \leqslant \delta_2$ and (6) give $\|(2-\lambda)f(x)\| \leqslant 5\delta$. Hence $\|f(x)\| \leqslant 5\delta$.
(ii) We observe that

$$\begin{aligned} &\|f(xy^{-2}) + (2-\lambda^2)f(x) + f(xy^2)\| \\ &\leqslant \mathscr{F}_y^{(\lambda)}(xy^{-1}) + \lambda\mathscr{F}_y^{(\lambda)}(x) + \mathscr{F}_y^{(\lambda)}(xy) \\ &\leqslant (2+|\lambda|)\delta. \end{aligned} \tag{7}$$

Consider the alternatives in $\mathscr{S} f_{y^2}^{(\lambda)}(x)$. The inequality $\mathscr{F}_{y^2}^{(2)}(x) \leqslant \delta_1$ and (7) give $\|(4 - \lambda^2)f(x)\| \leqslant (3 + |\lambda|)\delta$, while the inequality $\mathscr{F}_{y^2}^{(\lambda)}(x) \leqslant \delta_2$ and (7) give $\|(2+\lambda-\lambda^2)f(x)\| \leqslant (3+|\lambda|)\delta$. Hence $\|f(x)\| \leqslant (3+|\lambda|)\delta$. □

Lemma 2 *Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$. If $\mathscr{F}_y^{(2)}(xy^{-1}) \leqslant \delta_1$, $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$, and $\mathscr{F}_y^{(\lambda)}(xy) \leqslant \delta_2$, then $\|f(x)\| \leqslant \big(14 + 14|\lambda| + 12\lambda^2 + 4|\lambda|^3\big)\delta$.*

Proof: By $\mathscr{F}_y^{(2)}(xy^{-1}) \leqslant \delta_1$ and $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$, we obtain

$$\|f(xy^{-2}) + (1-2\lambda)f(x) + 2f(xy)\| \leqslant 3\delta. \tag{8}$$

Next, we consider two possible cases in $\mathscr{S} f_{y^2}^{(\lambda)}(x)$:

Case (i): Assume that $\mathscr{F}_{y^2}^{(2)}(x) \leqslant \delta_1$. Using $\mathscr{F}_y^{(\lambda)}(xy) \leqslant \delta_2$, $\mathscr{F}_{y^2}^{(2)}(x) \leqslant \delta_1$, and (8) we obtain

$$\|2f(x) + f(xy)\| \leqslant 5\delta \tag{9}$$

and

$$\|(2\lambda+1)f(x) + f(xy^2)\| \leqslant (2 + 4|\lambda|)\delta. \tag{10}$$

Eliminating $f(xy)$ from (9) and the alternatives in $\mathscr{S} f_y^{(\lambda)}(xy^2)$, we have

$$\begin{aligned} &\|2f(x) + 2f(xy^2) - f(xy^3)\| \leqslant 6\delta \quad \text{or} \\ &\|2f(x) + \lambda f(xy^2) - f(xy^3)\| \leqslant 6\delta. \end{aligned} \tag{11}$$

By (10) and (11), we obtain

$$\begin{aligned} &\|4\lambda f(x) + f(xy^3)\| \leqslant (10 + 8|\lambda|)\delta \quad \text{or} \\ &\|(2\lambda^2 + \lambda - 2)f(x) + f(xy^3)\| \leqslant (6 + 2|\lambda| + 4\lambda^2)\delta. \end{aligned} \tag{12}$$

Consider the alternatives in $\mathscr{S} f_{y^2}^{(\lambda)}(xy)$.
(i) If $\mathscr{F}_{y^2}^{(2)}(xy) \leqslant \delta_1$, then we use $\mathscr{F}_{y^2}^{(2)}(xy) \leqslant \delta_1$ and $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$ to obtain

$$\|\lambda f(x) - 3f(xy) + f(xy^3)\| \leqslant 2\delta. \tag{13}$$

By (12) and (13), we obtain

$$\begin{aligned} &\|3\lambda f(x) + 3f(xy)\| \leqslant (12 + 8|\lambda|)\delta \quad \text{or} \\ &\|(2\lambda^2 - 2)f(x) + 3f(xy)\| \leqslant (8 + 2|\lambda| + 4\lambda^2)\delta. \end{aligned} \tag{14}$$

Eliminating $f(xy)$ from (9) and (14), we have $\|f(x)\| \leqslant (12 + |\lambda| + 2\lambda^2)\delta$.
(ii) If $\mathscr{F}_{y^2}^{(\lambda)}(xy) \leqslant \delta_2$, then we use $\mathscr{F}_{y^2}^{(\lambda)}(xy) \leqslant \delta_2$ and $\mathscr{F}_y^{(\lambda)}(x) \leqslant \delta_2$ to obtain

$$\|\lambda f(x) - (\lambda+1)f(xy) + f(xy^3)\| \leqslant 2\delta. \tag{15}$$

By (12) and (15), we obtain

$$\begin{aligned} &\|3\lambda f(x) + (\lambda+1)f(xy)\| \leqslant (12 + 8|\lambda|)\delta \quad \text{or} \\ &\|(2\lambda^2 - 2)f(x) + (\lambda+1)f(xy)\| \leqslant (8 + 2|\lambda| + 4\lambda^2)\delta. \end{aligned} \tag{16}$$

Eliminating $f(xy)$ from (9) and (16), we obtain $\|f(x)\| \leqslant (17+11|\lambda|+2\lambda^2)\delta$.

Case (ii). Assume that $\mathscr{F}_{y^2}^{(\lambda)}(x) \leqslant \delta_2$. Using $\mathscr{F}_{y}^{(\lambda)}(xy) \leqslant \delta_2$, $\mathscr{F}_{y^2}^{(\lambda)}(x) \leqslant \delta_2$, and (8) we obtain

$$\|f(x)+f(xy)\| \leqslant 5\delta \tag{17}$$

and

$$\|(\lambda+1)f(x)+f(xy^2)\| \leqslant (2+4|\lambda|)\delta. \tag{18}$$

Eliminating $f(xy^2)$ from (18) and the alternatives in $\mathscr{S}f_{y^2}^{(\lambda)}(xy^2)$, we obtain

$$\begin{aligned} &\|(2\lambda+3)f(x)+f(xy^4)\| \leqslant (5+8|\lambda|)\delta \quad \text{or} \\ &\|(\lambda^2+\lambda+1)f(x)+f(xy^4)\| \leqslant (1+2|\lambda|+4\lambda^2)\delta. \end{aligned} \tag{19}$$

By (17) and the alternatives in $\mathscr{S}f_{y}^{(\lambda)}(xy^2)$, we have

$$\begin{aligned} &\|f(x)+2f(xy^2)-f(xy^3)\| \leqslant 6\delta \quad \text{or} \\ &\|f(x)+\lambda f(xy^2)-f(xy^3)\| \leqslant 6\delta. \end{aligned} \tag{20}$$

Consider the alternatives in $\mathscr{S}f_{y}^{(\lambda)}(xy^3)$ as follows. (i) If $\mathscr{F}_{y}^{(2)}(xy^3) \leqslant \delta_1$, then we eliminate $f(xy^3)$ from (20) and $\mathscr{F}_{y}^{(2)}(xy^3) \leqslant \delta_1$ to obtain

$$\begin{aligned} &\|2f(x)+3f(xy^2)-f(xy^4)\| \leqslant 13\delta \quad \text{or} \\ &\|2f(x)+(2\lambda-1)f(xy^2)-f(xy^4)\| \leqslant 13\delta. \end{aligned} \tag{21}$$

Using (18) and (21), we obtain

$$\begin{aligned} &\|(3\lambda+1)f(x)+f(xy^4)\| \leqslant (19+12|\lambda|)\delta \quad \text{or} \\ &\|(2\lambda^2+\lambda-3)f(x)+f(xy^4)\| \leqslant (15+8|\lambda|+8\lambda^2)\delta. \end{aligned} \tag{22}$$

By (19) and (22), we conclude that

$$\|f(x)\| \leqslant (24+12|\lambda|+12\lambda^2)\delta.$$

(ii) If $\mathscr{F}_{y}^{(\lambda)}(xy^3) \leqslant \delta_2$, then we eliminate $f(xy^3)$ from (20) and $\mathscr{F}_{y}^{(\lambda)}(xy^3) \leqslant \delta_2$ to obtain

$$\begin{aligned} &\|\lambda f(x)+(2\lambda-1)f(xy^2)-f(xy^4)\| \leqslant (1+6|\lambda|)\delta \text{ or} \\ &\|\lambda f(x)+(\lambda^2-1)f(xy^2)-f(xy^4)\| \leqslant (1+6|\lambda|)\delta. \end{aligned} \tag{23}$$

Using (18) and (23), we obtain

$$\begin{aligned} &\|(2\lambda^2-1)f(x)+f(xy^4)\| \leqslant (3+14|\lambda|+8\lambda^2)\delta \quad \text{or} \\ &\|(\lambda^3+\lambda^2-2\lambda-1)f(x)+f(xy^4)\| \\ &\qquad \leqslant (3+10|\lambda|+2\lambda^2+4|\lambda^3|)\delta. \end{aligned} \tag{24}$$

By (19) and (24), we conclude that

$$\|f(x)\| \leqslant (8+14|\lambda|+12\lambda^2+4|\lambda^3|)\delta 4.$$

The desired bound of $f(x)$ follows from the consideration of all cases. □

The following lemma is crucial for the main theorem in the next section.

Lemma 3 *If $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$, then $\mathscr{F}_{y}^{(2)}(x) \leqslant \mathscr{M}_{\delta}^{\lambda}$ for all $x, y \in G$.*

Proof: Let $f \in \mathscr{A}_{(G,E)}^{(\lambda)}$ and $x, y \in G$. Suppose $\mathscr{F}_{y}^{(2)}(x) > \delta_1$. From the alternatives in $\mathscr{S}f_{y}^{(\lambda)}(x)$, we obtain $\mathscr{F}_{y}^{(\lambda)}(x) \leqslant \delta_2$. The alternatives in $\mathscr{S}f_{y}^{(\lambda)}(xy^{-1})$ will be considered as follows.

Case (i). Assume that $\mathscr{F}_{y}^{(2)}(xy^{-1}) \leqslant \delta_1$. By Lemma 1 and Lemma 2, we conclude that

$$\|f(x)\| \leqslant (14+14|\lambda|+12\lambda^2+4|\lambda^3|)\delta. \tag{25}$$

Using $\mathscr{F}_{y}^{(\lambda)}(x) \leqslant \delta_2$ and (3), we obtain

$$\begin{aligned} &\|f(xy^{-1})+f(xy)\| \\ &\qquad \leqslant (1+14|\lambda|+14\lambda^2+12|\lambda^3|+4\lambda^4)\delta. \end{aligned} \tag{26}$$

Hence, by (25) and (26), we have $\mathscr{F}_{y}^{(2)}(x) \leqslant \mathscr{M}_{\delta}^{\lambda}$ as desired.

Case (ii). Assume that $\mathscr{F}_{y}^{(\lambda)}(xy^{-1}) \leqslant \delta_2$. Consider the alternatives in $\mathscr{S}f_{y}^{(\lambda)}(xy)$. If $\mathscr{F}_{y}^{(\lambda)}(xy) \leqslant \delta_2$, then Lemma 1 gives $\|f(x)\| \leqslant (3+|\lambda|)\delta$. Thus the desired proof is similar to the above case. If $\mathscr{F}_{y}^{(2)}(xy) \leqslant \delta_1$, then the proof is as in case (i) after replacing y by y^{-1} and x by xy^{-1}. □

HYERS-ULAM STABILITY

It should be remarked that Srisawat et al[8] proved that when $\lambda \notin \{-2,-1,0\}$, the alternative Jensen's functional equation (4) is equivalent to Jensen's functional equation (3). On the other hand, when $\lambda \in \{-2,-1,0\}$, (4) is not necessarily equivalent to (3). In this section, we will prove the Hyers-Ulam stability of the alternative Jensen's functional equation (4) when $\lambda \notin \{-2,-1,0,2\}$ is an integer by the so-called *direct* method. The stability results of Jensen's functional equation can be found in, for instance, Kominek[9] or Jung[10].

Theorem 1 *Let $\tilde{G}$ be an abelian group. If $f \in \mathscr{A}_{(\tilde{G},E)}^{(\lambda)}$, then there exists a unique Jensen's mapping $J : \tilde{G} \to E$ satisfying (3) with $J(0) = f(0)$ such that $\|f(x) - J(x)\| \leqslant 2\mathscr{M}_{\delta}^{\lambda}$ for all $x \in \tilde{G}$. Furthermore, the mapping J is given by*

$$J(x) = f(0) + \lim_{n\to\infty} \tfrac{1}{2^n}(f(x^{2^n}) - f(0))$$

for all $x \in \tilde{G}$.

Proof: Assume that $f \in \mathscr{A}_{(\tilde{G},E)}^{(\lambda)}$. By Lemma 3, we obtain $\mathscr{F}_{y}^{(1)}(x) \leqslant \mathscr{M}_{\delta}^{\lambda}$ for all $x, y \in \tilde{G}$, i.e.,

$$\|f(xy^{-1}) - 2f(x) + f(xy)\| \leqslant \mathscr{M}_{\delta}^{\lambda}.$$

We define a function $\tilde{f} : \tilde{G} \to E$ by $\tilde{f}(x) = f(x) - f(0)$. It can be observed that $\tilde{f}(0) = 0$. Then for each $x, y \in \tilde{G}$, we have

$$\left\| \tfrac{1}{2}\big(\tilde{f}(xy) + \tilde{f}(xy^{-1})\big) - \tilde{f}(x) \right\| \leqslant \tfrac{1}{2}\mathscr{M}_\delta^\lambda. \tag{27}$$

Putting $y = x$, we obtain

$$\left\| \tfrac{1}{2}\tilde{f}(x^2) - \tilde{f}(x) \right\| \leqslant \tfrac{1}{2}\mathscr{M}_\delta^\lambda. \tag{28}$$

For each positive integer n and each $x \in \tilde{G}$, we apply (28) to obtain

$$\begin{aligned} \left\| \tfrac{1}{2^n}\tilde{f}(x^{2^n}) - \tilde{f}(x) \right\| &= \left\| \sum_{i=1}^{n} \left(\frac{\tilde{f}(x^{2^i})}{2^i} - \frac{\tilde{f}(x^{2^{i-1}})}{2^{i-1}} \right) \right\| \\ &\leqslant \left(1 - \tfrac{1}{2^n}\right) \mathscr{M}_\delta^\lambda. \end{aligned} \tag{29}$$

Consider the sequence $\{2^{-n} f(x^{2^n})\}$. For all positive integers m, n and every $x \in \tilde{G}$, we use (29) to obtain

$$\begin{aligned} \left\| \frac{\tilde{f}(x^{2^{n+m}})}{2^{n+m}} - \frac{\tilde{f}(x^{2^n})}{2^n} \right\| &= \tfrac{1}{2^n} \left\| \frac{\tilde{f}(x^{2^n \cdot 2^m})}{2^m} - \tilde{f}(x^{2^n}) \right\| \\ &\leqslant \tfrac{1}{2^n} \left(1 - \tfrac{1}{2^m}\right) \mathscr{M}_\delta^\lambda. \end{aligned}$$

Hence $\{2^{-n} f(x^{2^n})\}$ is a Cauchy sequence. We can define a function $\tilde{J} : \tilde{G} \to E$ by

$$\tilde{J}(x) = \lim_{n\to\infty} \frac{\tilde{f}(x^{2^n})}{2^n}.$$

Replacing x by x^{2^n} and y by y^{2^n} in (27), we obtain

$$\left\| \tfrac{1}{2}\big(\tilde{f}(x^{2^n} y^{2^n}) + \tilde{f}(x^{2^n} y^{-2^n})\big) - \tilde{f}(x^{2^n}) \right\| \leqslant \tfrac{1}{2}\mathscr{M}_\delta^\lambda. \tag{30}$$

Next, multiplying (30) by 2^{-n} and taking $n \to \infty$, we obtain

$$\tilde{J}(xy) + \tilde{J}(xy^{-1}) - 2\tilde{J}(x) = 0.$$

From (29), as $n \to \infty$, we have

$$\|\tilde{f}(x) - \tilde{J}(x)\| \leqslant \mathscr{M}_\delta^\lambda$$

for all $x \in G$. To show the uniqueness of $\tilde{J}$, let $\mathscr{J} : \tilde{G} \to E$ satisfy $\mathscr{J}(0) = 0$ and $\|\tilde{f}(x) - \mathscr{J}(x)\| \leqslant \mathscr{M}_\delta^\lambda$ for all $x \in \tilde{G}$. For every positive integer n, we obtain

$$\tilde{J}(x^{2^n}) = 2^n \tilde{J}(x), \quad \mathscr{J}(x^{2^n}) = 2^n \mathscr{J}(x).$$

Hence

$$\begin{aligned} &\|\mathscr{J}(x) - \tilde{J}(x)\| \\ &\quad= \left\| \tfrac{1}{2^n}\big(\tilde{J}(x^{2^n}) - \tilde{f}(x^{2^n})\big) - \tfrac{1}{2^n}\big(\tilde{f}(x^{2^n}) - \mathscr{J}(x^{2^n})\big) \right\| \\ &\quad\leqslant \tfrac{1}{2^n} \left\| \tilde{f}(x^{2^n}) - \tilde{J}(x^{2^n}) \right\| + \tfrac{1}{2^n} \left\| \tilde{f}(x^{2^n}) - \mathscr{J}(x^{2^n}) \right\| \\ &\quad\leqslant \tfrac{2}{2^n}\mathscr{M}_\delta^\lambda. \end{aligned} \tag{31}$$

As $n \to \infty$ in (31), we have $\mathscr{J}(x) = \tilde{J}(x)$ for all $x \in \tilde{G}$. By defining a function $J : \tilde{G} \to E$ by $J(x) = \tilde{J}(x) + f(0)$ for all $x \in \tilde{G}$, the proof is complete. □

REFERENCES

1. Kannappan PL, Kuczma M (1974) On a functional equation related to the Cauchy equation. *Ann Polon Math* **30**, 49–55.
2. Ger R (1977) On an alternative functional equation. *Aequationes Mathematicae* **15**, 145–162.
3. Forti GL (1979) La soluzione generale dell'equazione funzionale $\{cf(x+y) - af(x) - bf(y) - d\}\{f(x+y) - f(x) - f(y)\} = 0$. *Matematiche (Catania)* **34**, 219–242.
4. Nakmahachalasint P (2012) An alternative Jensen's functional equation on semigroups. *ScienceAsia* **38**, 408–413.
5. Ng CT (1990) Jensen's functional equation on groups. *Aequationes Mathematicae* **39**, 85–99.
6. Parnami JC, Vasudeva HL (1992) On Jensen's functional equation. *Aequationes Mathematicae* **43**, 211–218.
7. Nakmahachalasint P (2013) Stability of an alternative Jensen's functional equation. *ScienceAsia* **39**, 643–648.
8. Srisawat C, Kitisin N, Nakmahachalasint P (2015) On an alternative functional equation of Jensen type on groups. *ScienceAsia* **41**, 280–288.
9. Kominek Z (1989) On a local stability of the Jensen functional equation. *Demonstratio Math* **22**, 499–507.
10. Jung SM (1998) Hyers-Ulam-Rassias stability of Jensen's equation and its application. *Proc Am Math Soc* **126**, 3137–3143.