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The average binding number of graphs

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ABSTRACT: The binding number is a measure of the vulnerability of a graph. We investigate a refinement that involves the average of this parameter. Like the binding number itself, the average binding number $\mathrm{bind}_{\mathrm{av}}(G)$ of G measures the vulnerability of a graph, which is more sensitive. In this study, some bounds of the average binding number of some special graphs are obtained. Furthermore some results about the average binding number of graphs obtained from graph operations are also provided.

KEYWORDS: graph vulnerability, network design and communication, connectivity, average lower connectivity

MSC2010: 05C40 68M10 68R10

INTRODUCTION

One of the most important problems which is solved by the help of graph theory is to design a network model whose resistance for the disruptions is more than other networks. Graphs are often used to model real world problems such as in a communication, computer, or spy network. In a network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or links. Parameters used to measure the vulnerability of networks include connectivity[4] average lower connectivity[1], binding number[13], etc.

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of a vertex $u \in V(G)$ is the set $N(u) := \{v \in V(G) \mid uv \in E(G)\}$. This is more commonly called the open neighbourhood of a vertex. The neighbourhood of a set $S \subseteq V(G)$ is $N(S) = \cup_{u \in S} N(u)$. Let $d(u)$ denote the degree of the vertex u in G. The maximum degree of a graph G is the largest vertex degree of G, denoted $\Delta(G)$, and similarly, the minimum vertex degree is the smallest vertex degree of G, denoted $\delta(G)$. The distance $d(u,v)$ between two vertices u and v of a finite graph is the minimum length of the paths connecting them, i.e., the length of a graph geodesic. The diameter d of a graph is the greatest distance between any pair of vertices. A minimum vertex cover is a vertex cover having the smallest possible number of vertices for a given graph. The size of a minimum vertex cover of a graph G is known as the vertex cover number and is denoted $\alpha(G)$. An independent vertex set of a graph G is a subset of the vertices such that no two vertices in the subset represent an edge of G. Given a vertex cover of a graph, all vertices not in the cover define a independent vertex set. A maximum independent vertex set is an independent vertex set containing the largest possible number of vertices for a given graph. The independence number of G, denoted by $\beta(G)$, is the number of vertices in a maximum independent set of G. A spanning subgraph of G is a subgraph that contains every vertex of G.

Aslan[1] introduced the concept of average lower connectivity. For a vertex v of a graph G, the lower connectivity at v, denoted $s_v(G)$, is the smallest number of vertices in a set that contains v whose deletion from G produces a disconnected or a trivial graph. The average lower connectivity denoted by $\kappa_{\mathrm{av}}(G)$, is the value $\Sigma_{v \in V(G)} s_v(G)/n$, where n denotes the number of vertices in the graph G and $\Sigma_{v \in V(G)} s_v(G)$ denotes the sum over all vertices of G.

Woodall[13] defined the binding number of a graph G as

$$\mathrm{bind}(G) = \min_{S \in F(G)} \left\{ \frac{|N(S)|}{|S|} \right\},$$

where $F(G) = \{S \subseteq V(G) \mid S \neq \varnothing, N(S) \neq V(G)\}$. A binding set of G is any set S such that $\mathrm{bind}(G) = |N(S)|/|S|$.

The study of binding number in graphs is an important research area, perhaps also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical

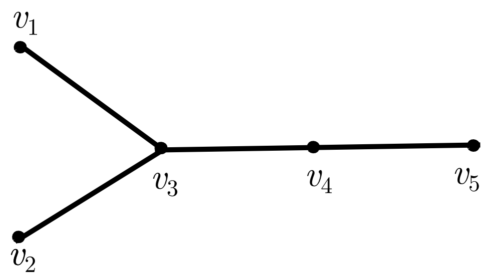

Fig. 1 The graph G.

and real world problems. More recently interest has been rearoused, yielding a succession of results covering inter alia product graphs[7,11,12], cliques and cycles as subgraphs, and the range of possible values of binding numbers[8]. Linzhung et al[9] have extended the binding number to the edges and the studied the edge-binding number of some plane graph.

The average parameters have been found to be more useful in some circumstances than the corresponding measures based on worst-case situations[2,3,6,10]. Thus incorporating the concept of the binding number and the idea of the average lower connectivity introduces a new graph parameter called the average binding number, $\mathrm{bind}_{\mathrm{av}}(G)$.

For $v \in V(G)$, the local binding number of v is

$$\mathrm{bind}_v(G) = \min_{S \in F_v(G)} \left\{ \frac{|N(S)|}{|S|} \right\},$$

where $F_v(G) = \{S \subseteq V(G) \mid v \in S, S \neq \emptyset, N(S) \neq V(G)\}$. Clearly,

$$\mathrm{bind}(G) = \min_{v \in V(G)} \{\mathrm{bind}_v(G)\}.$$

A local binding set of v in G is $S \in F_v(G)$ such that $\mathrm{bind}_v(G) = |N(S)|/|S|$. Furthermore, the average binding number of G is defined as

$$\mathrm{bind}_{\mathrm{av}}(G) = \frac{1}{n} \sum_{v \in V(G)} \mathrm{bind}_v(G),$$

where n is the number of vertices in graph G.

Example 1 Consider the graph G in Fig. 1, where $|V(G)| = 5$ and $|E(G)| = 4$. Note that $\mathrm{bind}_{v_1} = \frac{1}{2}$, $\mathrm{bind}_{v_2} = \frac{1}{2}$, $\mathrm{bind}_{v_3} = \frac{4}{4} = 1$, $\mathrm{bind}_{v_4} = \frac{2}{3}$, and $\mathrm{bind}_{v_5} = \frac{2}{3}$. It follows that

$$\mathrm{bind}_{\mathrm{av}}(G) = \frac{1}{5}\left(\frac{1}{2} + \frac{1}{2} + 1 + \frac{2}{3} + \frac{2}{3}\right) = \frac{2}{3}.$$

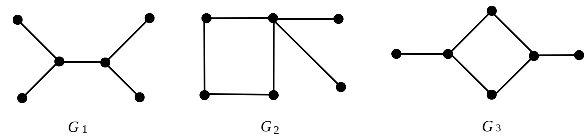

Fig. 2 The graphs G_1, G_2 and G_3.

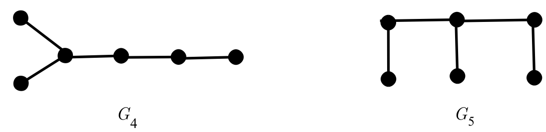

Fig. 3 The graphs G_4 and G_5.

The following examples show that the average binding number is more efficient than the connectivity, the average lower connectivity, and the binding number in measuring the vulnerability of some graphs.

Example 2 It is easy to see that connectivity of a star $K_{1,4}$ and a path P_5 are equal,

$$\kappa(K_{1,4}) = \kappa(P_5) = 1.$$

however, the average binding number of a star $K_{1,4}$ and a path P_5 are different,

$$\mathrm{bind}_{\mathrm{av}}(K_{1,4}) = 1, \quad \mathrm{bind}_{\mathrm{av}}(P_5) = \frac{14}{15}.$$

Example 3 Let G_1, G_2, and G_3 be the graphs in Fig. 2. It is easy to see that the connectivity and binding number of G_1, G_2, and G_3 are equal;

$$\kappa(G_1) = \kappa(G_2) = \kappa(G_3) = 1,$$
$$\mathrm{bind}(G_1) = \mathrm{bind}(G_2) = \mathrm{bind}(G_3) = \frac{1}{2}.$$

However, the average binding number of G_1, G_2, and G_3 are different,

$$\mathrm{bind}_{\mathrm{av}}(G_1) = \frac{3}{5}, \ \mathrm{bind}_{\mathrm{av}}(G_2) = \frac{81}{120}, \ \mathrm{bind}_{\mathrm{av}}(G_3) = \frac{2}{3}.$$

Example 4 Let G_4 and G_5 be the graphs in Fig. 3. It is easy to check that the average lower connectivity of G_4 and G_5 are equal,

$$\kappa_{\mathrm{av}}(G_4) = \kappa_{\mathrm{av}}(G_5) = \frac{3}{2}.$$

However, the average binding number of G_1 and G_2 are different,

$$\mathrm{bind}_{\mathrm{av}}(G_4) = \frac{2}{3}, \quad \mathrm{bind}_{\mathrm{av}}(G_5) = 1.$$

BOUNDS FOR AVERAGE BINDING NUMBER

In this study, some bounds of the average binding number are obtained for some special graphs. Furthermore some results of the average binding number of graphs generated by graph operations are also given. The related theorems of the average binding number and other graph parameters are provided as the followings.

Theorem 1 *If G is a graph of order n with the minimum degree $\delta(G)$, then*

$$\mathrm{bind}_{\mathrm{av}}(G) \geqslant \frac{\delta(G)}{n-\delta(G)}.$$

Proof: Let $v \in V(G)$ and S_v be a local binding set at v. Certainly $|N(S_v)| \geqslant d(v) \geqslant \delta(G)$. Since $N(S_v) \neq V(G)$, S_v omit all the neighbours of some vertex w, and $|S_v| \leqslant n - d(w) \leqslant n - \delta(G)$. Thus

$$\mathrm{bind}_v(G) = \frac{|N(S_v)|}{|S_v|} \geqslant \frac{\delta(G)}{n-\delta(G)}.$$

Hence

$$\mathrm{bind}_{\mathrm{av}}(G) = \frac{1}{n}\sum_{v\in V(G)} \mathrm{bind}_v(G) \geqslant \frac{\delta(G)}{n-\delta(G)}.$$

□

Theorem 2 *If G is a graph of order n with covering number $\alpha(G)$ and independence number $\beta(G)$, then*

$$\mathrm{bind}_{\mathrm{av}}(G) \leqslant \frac{\alpha(G)(\beta(G)+n-1)}{n\beta(G)}.$$

Proof: Let $v \in V(G)$ and M be a maximum independent set of G.

If $v \in V(M)$, then there is a local binding set $S_v \in F_v(G)$ that contains all vertices in the maximum independent set of G, and $|S_v| = \beta(G)$. Then $|N(S_v)| = \alpha(G)$ and $\mathrm{bind}_v(G) = \alpha(G)/\beta(G)$.

If $v \notin V(M)$, then for $S \in F_v(G)$, $|S| \geqslant \beta(G)$ and $|N(S)| \leqslant n-1$. Thus $\mathrm{bind}_v(G) \leqslant (n-1)/\beta(G)$. Hence

$$\begin{aligned}\mathrm{bind}_{\mathrm{av}}(G) &\leqslant \frac{\beta(G)\left(\frac{\alpha(G)}{\beta(G)}\right) + \alpha(G)\left(\frac{n-1}{\beta(G)}\right)}{n}\\ &= \frac{\alpha(G)(\beta(G)+n-1)}{n\beta(G)}.\end{aligned}$$

□

Theorem 3 *If G is a graph of order n with the minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, then*

$$\mathrm{bind}_{\mathrm{av}}(G) \geqslant \frac{\delta(G)}{\Delta(G)+1}.$$

Proof: Let $v \in V(G)$. For $S_v \in F_v(G)$, $|S_v| \leqslant \Delta(G)+1$ and $|N(S_v)| \geqslant \delta(G)$, hence

$$\mathrm{bind}_v(G) = \min_{S_v\in F_v(G)} \frac{|N(S_v)|}{|S_v|} \geqslant \frac{\delta(G)}{\Delta(G)+1}.$$

Thus

$$\mathrm{bind}_{\mathrm{av}}(G) = \frac{1}{n}\sum_{v\in V(G)} \mathrm{bind}_v(G) \geqslant \frac{\delta(G)}{\Delta(G)+1}.$$

□

Theorem 3 implies that $\mathrm{bind}_{\mathrm{av}}(T) \geqslant 1/(\Delta(T)+1)$ for a tree T.

Theorem 4 *If H is a spanning subgraph of G, then*

$$\mathrm{bind}_{\mathrm{av}}(H) \leqslant \mathrm{bind}_{\mathrm{av}}(G).$$

Proof: Let $v \in V(G) \cap V(H)$ with a local binding set $S_v^* \in F_v(G)$ of G. Let denote $N_G(S_v^*) = N(S_v^*) \cap V(G)$ and $N_H(S_v^*) = N(S_v^*) \cap V(H)$. Then $N_H(S_v^*) \subseteq N_G(S_v^*)$ and

$$\mathrm{bind}_v(H) \leqslant \frac{|N_H(S_v^*)|}{|S_v^*|} \leqslant \frac{|N_G(S_v^*)|}{|S_v^*|} = \mathrm{bind}_v(G).$$

Thus by the definition, $\mathrm{bind}_{\mathrm{av}}(H) \leqslant \mathrm{bind}_{\mathrm{av}}(G)$. □

Theorem 5 *If G is a graph of order n, then*

$$\mathrm{bind}(G) \leqslant \mathrm{bind}_{\mathrm{av}}(G).$$

Proof: From the definitions it is clear that for $v \in V(G)$, $\mathrm{bind}(G) \leqslant \mathrm{bind}_v(G)$. Thus

$$\mathrm{bind}(G) \leqslant \frac{1}{n}\sum_{v\in V(G)} \mathrm{bind}_v(G) = \mathrm{bind}_{\mathrm{av}}(G).$$

□

AVERAGE BINDING NUMBER OF CLASSES OF GRAPHS

Theorem 6 *If P_n is a path of order $n \geqslant 3$, then*

$$\mathrm{bind}_{\mathrm{av}}(P_n) = \begin{cases} 1, & n \text{ even},\\ \frac{2n^2-5n+3}{2n^2-4n}, & n \text{ odd}.\end{cases}$$

Proof: Let $v \in V(P_n)$ and $S_v \in F_v(P_n)$.

Case 1: n is even. Since $|N(S_v)| \geqslant |S_v|$ with the equality holds when S_v is a maximum independent set of P_n, therefore, $\mathrm{bind}_v(P_n) = 1$ for all v, which implies $\mathrm{bind}_{\mathrm{av}}(P_n) = 1$.

Case 2: n is odd. Let the vertices of P_n be $p_1, p_2, \dots, p_n$ in order along the path. There is a maximum independent set that gives the unique independence number $\beta(P_n) = \frac{1}{2}(n+1)$.

For any v in the maximum independent set containing $\frac{1}{2}(n+1)$ vertices, the $\text{bind}_v(P_n)$ is obtained when $|S_v| = \beta(P_n) = \frac{1}{2}(n+1)$ and $|N(S_v)| = \frac{1}{2}(n-1)$. Thus $\text{bind}_v(P_n) = (n-1)/(n+1)$.

For any v not in a maximum independent set of $\frac{1}{2}(n-1)$ vertices, to obtain $\text{bind}_v(P_n)$ we require S_v as large as possible without $N(S_v)$ being the whole of $V(P_n)$, i.e., $|S_v| = n-2$, for example, $S_v = \{p_3, p_4, \dots, p_n\}$). Thus $\text{bind}_v(P_n) = (n-1)/(n-2)$.

Hence, by the definition

$$\begin{aligned}\text{bind}_{\text{av}}(P_n) &= \frac{1}{n}\left(\frac{(n+1)(n-1)}{2(n+1)} + \frac{(n-1)(n-1)}{2(n-2)}\right)\\ &= \frac{2n^2-5n+3}{2n^2-4n}.\end{aligned}$$

□

Theorem 7 *If C_n is a cycle of order $n \geqslant 4$, then*

$$\text{bind}_{\text{av}}(C_n) = \begin{cases} 1, & n \text{ even},\\ \frac{n-1}{n-2}, & n \text{ odd}.\end{cases}$$

Proof: Let $v \in V(C_n)$ and $S_v \in F_v(C_n)$.

Case 1: n is even. For all $v \in V(C_n)$, if $|S_v| = r$ then $|N(S_v)| \geqslant r$, therefore, $\text{bind}_v(C_n) \geqslant 1$. Since there is a local binding set S_v^* of C_n such that $|S_v^*| = \frac{1}{2}n$, when S_v^* is a maximum independent set of C_n, and $|N(S_v^*)| = \frac{1}{2}n$. Hence $\text{bind}_v(C_n) = 1$, for $v \in V(C_n)$, and therefore, $\text{bind}_{\text{av}}(C_n) = 1$

Case 2: n is odd. For $|S_v| = r$, then $r \leqslant n - \delta(C_n) = n-2$ and $|N(S_v)| \geqslant r+1$. Thus $|N(S_v)|/|S_v| \geqslant (r+1)/r$. is a decreasing function of r that has the minimum value when $r = n-2$. Hence $\text{bind}_v(C_n) \geqslant (n-1)/(n-2)$. Since there is a binding set S_v^* of C_n such that $|S_v^*| = n-2$ and $|N(S_v)| = n-1$, thus $\text{bind}_v(C_n) = (n-1)/(n-2)$, and therefore, $\text{bind}_{\text{av}}(C_n) = (n-1)/(n-2)$. □

Theorem 8 *If K_n is a complete graph of order $n \geqslant 2$, then* $\text{bind}_{\text{av}}(K_n) = n-1$.

Proof: Let $v \in V(K_n)$ and $S_v \in F_v(K_n)$. If $|S_v| \geqslant 2$, then $N(S_v) = V(K_n)$, which contradicts with $S_v \in F_v(K_n)$. Thus $|S_v| = 1$ and $|N(S_v)| = n-1$. Hence $\text{bind}_v(K_n) = n-1$ for all $v \in V(K_n)$ and $\text{bind}_{\text{av}}(K_n) = n-1$. □

Theorem 9 *If $K_{a,b}$ is a complete bipartite graph of order $a+b$ with $1 \leqslant a \leqslant b$, then*

$$\text{bind}_{\text{av}}(K_{a,b}) = 1.$$

Proof: Let $V(K_{a,b}) = V(G_1) \cup V(G_2)$ be the vertex set of $K_{a,b}$, where the set $V(G_1)$ contains a vertices having degree b and the set $V(G_2)$ contains b vertices having degree a. For $v \in V(K_{a,b})$ and $S_v \in F_v(V(K_{a,b}))$, if $|S_v \cap V(G_1)| \neq 0$ and $|S_v \cap V(G_2)| \neq 0$ then $N(S_v) = V(K_{a,b})$, a contradiction. This implies that $S_v \subseteq V(G_1)$ to have $N(S_v) = V(G_2)$, or $S_v \subseteq V(G_2)$ to have $N(S_v) = V(G_1)$. Since for a vertices of $v \in V(G_1)$, $|S_v| \leqslant a$ and $|N(S_v)| = b$, i.e., $\text{bind}_v(K_{a,b}) = b/a$, and for b vertices of $v \in V(G_2)$, $|S_v| \leqslant b$ and $|N(S_v)| = a$, i.e., $\text{bind}_v(K_{a,b}) = a/b$. Therefore,

$$\text{bind}_{\text{av}}(K_{a,b}) = \frac{1}{a+b}\left(a\frac{b}{a} + b\frac{a}{b}\right) = 1.$$

□

Corollary 1 *If $K_{1,n}$ is a star graph, then*

$$\text{bind}_{\text{av}}(K_{1,n}) = 1.$$

GRAPH OPERATIONS

This section provides some results of the average binding number of graphs obtained from graph operators.

Power of a graph

Definition 1 [Ref. 5] The kth power, G^k, of a connected graph G is the graph with $V(G^k) = V(G)$ for which $uv \in E(G^k)$ if $1 \leqslant d(u,v) \leqslant k$.

Theorem 10 *If G is a graph of order n and diameter d, then*

$$\text{bind}_{\text{av}}(G) \leqslant \text{bind}_{\text{av}}(G^2) \leqslant \cdots \leqslant \text{bind}_{\text{av}}(G^d) = n-1.$$

Proof: Since for positive integer i, G^i is a subgraph of G^{i+1}, it follows from Theorem 4 that

$$\text{bind}_{\text{av}}(G) \leqslant \text{bind}_{\text{av}}(G^2) \leqslant \cdots \leqslant \text{bind}_{\text{av}}(G^d).$$

Since G is connected with diameter d, then G^d is a complete graph, thus $\text{bind}_{\text{av}}(G^d) = n-1$. □

Join of graphs

Definition 2 [Ref. 5] The join $G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph consists of G_1, G_2, and all edges joining $V(G_1)$ and $V(G_2)$.

For a join graph $G+H$, it is easy to see that

$$\begin{aligned}\sum_{v\in V(G+H)} \text{bind}_v(G+H) = &\sum_{v\in V(G)} \text{bind}_v(G+H)\\ &+ \sum_{v\in V(H)} \text{bind}_v(G+H).\end{aligned}$$

Theorem 11 *Let G and H be two connected graphs of order m and n, respectively. Then*

$$\mathrm{bind}_{\mathrm{av}}(G+H) \geqslant \frac{m\cdot \mathrm{bind}(G)+n\cdot \mathrm{bind}(H)}{m+n}.$$

Proof: Let $v \in V(G+H)$ and $S_v \in F_v(G+H)$. Let denote $N_G(S_v)=N(S_v)\cap V(G)$ and $N_H(S_v)=N(S_v)\cap V(H)$. Since every vertex of G is connected to all vertices of H, and vice versa, then $N(S_v)=N_G(S_v)\cup V(H)$ or $N(S_v)=N_H(S_v)\cup V(G)$. Since $N(S_v)\neq V(G+H)$, $v\in V(G)$ implies $S_v\subseteq V(G)$, or $v\in V(H)$ implies $S_v\subseteq V(H)$.

For $v\in V(G)$, let A_v^* be its local binding set in G. Since $S_v\subseteq V(G)$, $S_v\in F_v(G+H)\cap F_v(G)$, thus

$$\frac{|N(S_v)|}{|S_v|}=\frac{|N_G(S_v)|+|V(H)|}{|S_v|}\geqslant\frac{|N_G(A_v^*)|}{|A_v^*|}+\frac{|V(H)|}{|S_v|},$$

$$\frac{|N(S_v)|}{|S_v|}=\mathrm{bind}_v(G)+\frac{|V(H)|}{|S_v|}\geqslant \mathrm{bind}(G). \tag{1}$$

Similarly, for $v\in V(H)$,

$$\frac{|N(S_v)|}{|S_v|}\geqslant \mathrm{bind}(H). \tag{2}$$

Hence by (1) and (2),

$$\mathrm{bind}_{\mathrm{av}}(G+H) \geqslant \frac{m\cdot \mathrm{bind}(G)+n\cdot \mathrm{bind}(H)}{m+n}.$$

□

Lemma 1 *Let G and H be two connected graphs. If $v\in V(G)$ or $v\in V(H)$ and S_v is local binding set of $G+H$, then S_v is either a local binding set of G or H.*

Proof: Suppose that $v\in V(G)$. Let $N_G(S_v)=N(S_v)\cap V(G)$. Since all vertices of G is connected to all vertices of H, $N(S_v)=N_G(S_v)\cup V(H)$, which implies that $S_v\subset V(G)$. Hence S_v is a local binding set of G. Similarly, if $v\in V(H)$, then S_v is a local binding set of H. □

Theorem 12 *Let m and n be positive integers. Then*

$$\mathrm{bind}_{\mathrm{av}}(K_m+P_n) = \begin{cases} m+1, & n \text{ even},\\ \frac{2n^2m+2m^2n+2n^2-4m^2-3mn-5n-m+3}{2n^2+2mn-4m-4n}, & n \text{ odd}.\end{cases}$$

Proof: Let $v\in V(K_m+P_n)$ and $S_v\in F_v(K_m+P_n)$. By Lemma 1, if $S_v\cap V(K_m)\neq\varnothing$ and $S_v\cap V(P_n)\neq\varnothing$, then $N(S_v)=V(K_m+P_n)$, a contradiction. Hence either $S_v\subseteq V(K_m)$ or $S_v\subseteq V(P_n)$.

If $v\in V(K_m)$, then $|S_v|=1$ otherwise $N(S_v)=V(K_m+P_n)$. Thus $\mathrm{bind}_v(K_m+P_n)=(m+n-1)$ and

$$\sum_{v\in V(K_m)}\mathrm{bind}_v(K_m+P_n)=m(m-1+n).$$

If $v\in V(P_n)$, we consider two cases.

Case 1: n is even. There is a local binding set S_v of P_n such that $|S_v|=n/2$, when S_v contains all vertices in a maximum independent set of P_n. Then $|N(S_v)|=m+n/2$ and $\mathrm{bind}_v(K_m+P_n)=(n+2m)/n$. Thus

$$\sum_{v\in V(P_n)}\mathrm{bind}_v(K_m+P_n)=n\left(\frac{n+2m}{n}\right)=n+2m.$$

Hence

$$\mathrm{bind}_{\mathrm{av}}(K_m+P_n)=\frac{m(m-1+n)+n+2m}{m+n} =\frac{(m+1)(m+n)}{m+n}=m+1.$$

Case 2: n is odd. If v is in a maximum independent set of P_n, then there is a local binding set $S_v\subset V(P_n)$ such that $|S_v|=\beta(P_n)=(n+1)/2$ and $|N(S_v)|=m+(n-1)/2$. Thus $\mathrm{bind}_v(K_m+P_n)=(2m+n-1)/(n+1)$ for these $(n+1)/2$ vertices v in a maximum independent set of P_n.

If v is not in a maximum independent set of P_n. To obtain $\mathrm{bind}_v(P_n)$, we require S_v as large as possible that $N(S_v)\neq V(P_n)$, when $|S_v|=n-2$, e.g., $S_v=\{p_3,p_4,\dots,p_n\}$, and $|N(S_v)|=n-1+m$. Thus $\mathrm{bind}_v(K_m+P_n)=(n-1+m)/(n-2)$ for these $\frac{1}{2}(n-1)$ vertices. Therefore

$$\sum_{v\in V(P_n)}\mathrm{bind}_v(K_m+P_n) =\frac{n+1}{2}\left(\frac{2m+n-1}{n+1}\right)+\frac{n-1}{2}\left(\frac{n-1+m}{n-2}\right) =\frac{2n^2+3mn-5n-5m+3}{2n-4}.$$

This implies that

$$\sum_{v\in V(K_m+P_n)}\mathrm{bind}_v(K_m+P_n) =m(m-1+n)+\frac{2n^2+3mn-5n-5m+3}{2n-4} =\frac{2n^2m+2m^2n+2n^2-4m^2-3mn-5n-m+3}{2n-4}.$$

Hence

$$\mathrm{bind}_{\mathrm{av}}(K_m+P_n)=\frac{\sum_{v\in V(K_m+P_n)}\mathrm{bind}_v(K_m+P_n)}{m+n}$$
$$=\frac{2n^2m+2m^2n+2n^2-4m^2-3mn-5n-m+3}{2n^2+2mn-4m-4n}.$$

The proof is completed. □

Theorem 13 *Let m and n be positive integers. Then*

$$\mathrm{bind}_{\mathrm{av}}(K_m+C_n)=\begin{cases} m+1, & n \text{ even},\\ \frac{n^2m+m^2n-2m^2+n^2-2mn+2m-n}{n^2+mn-2m-2n}, & n \text{ odd}.\end{cases}$$

Proof: Let $G=K_m+C_n$, $v\in V(G)$, and $S_v\in F_v(G)$. If $S_v\cap V(K_m)\neq\varnothing$ and $S_v\cap V(C_n)\neq\varnothing$, then $N(S_v)=V(G)$, a contradiction. Thus $S_v\subseteq V(K_m)$ or $S_v\subseteq V(C_n)$.

If $v\in V(K_m)$, then $|S_v|=1$, $|N(S_v)|=m-1+n$, and $\mathrm{bind}_v(G)=m-1+n$. Thus

$$\sum_{v\in V(K_m)}\mathrm{bind}_v(G)=m(m-1+n).$$

If $v\in V(C_n)$, we consider two cases.

Case 1: n is even. In this case there is a local binding set S_v^* of C_n and also G such that $|S_v^*|=n/2$ and $|N(S_v^*)|=m+n/2$. Thus $\mathrm{bind}_v(G)=(n+2m)/n$, and

$$\sum_{v\in V(C_n)}\mathrm{bind}_v(G)=n\left(\frac{n+2m}{n}\right)=n+2m.$$

Hence

$$\mathrm{bind}_{\mathrm{av}}(G)=\frac{m(m-1+n)+(n+2m)}{m+n}=m+1.$$

Case 2: n is odd. In this case there is a local binding set S_v^* of C_n and also G such that $|S_v^*|=n-2$, $|N(S_v^*)|=n-1+m$, and $\mathrm{bind}_v(G)=(n-1+m)/(n-2)$. Thus

$$\sum_{v\in V(C_n)}\mathrm{bind}_v(G)=n\left(\frac{n-1+m}{n-2}\right)=\frac{n^2-n+mn}{n-2}$$

and

$$\sum_{v\in V(G)}\mathrm{bind}_v(G)=m(m-1+n)+\frac{n^2-n+mn}{n-2}$$
$$=\frac{n^2m+m^2n-2m^2+n^2-2mn+2m-n}{n-2}.$$

Consequently,

$$\mathrm{bind}_{\mathrm{av}}(G)=\frac{n^2m+m^2n-2m^2+n^2-2mn+2m-n}{(m+n)(n-2)}$$
$$=\frac{n^2m+m^2n-2m^2+n^2-2mn+2m-n}{n^2+mn-2m-2n}.$$

The proof is completed. □

Corollary 2 *If W_n is a wheel graph order $n+1$, $n\geqslant 4$, then*

$$\mathrm{bind}_{\mathrm{av}}(W_n)=\begin{cases}2, & n \text{ even},\\ \frac{2n^2-2n}{n^2-n-2}, & n\text{ odd}.\end{cases}$$

Proof: Since $W_n=K_1+C_n$, Theorem 13 gives $\mathrm{bind}_{\mathrm{av}}(W_n)=\mathrm{bind}_{\mathrm{av}}(K_1+C_n)$ and the proof is completed. □

Corona of graphs

Definition 3 [Ref. 5] The corona $G\circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies H_i of H, and then joining the ith vertex of G to every vertex of H_i.

Theorem 14 *Let G and H be two connected graphs of order m and n, respectively, and A^* a binding set of H. Then*

$$\mathrm{bind}_{\mathrm{av}}(G\circ H)\geqslant\frac{(m-1)(n+1)+|N(A^*)|+1}{(m-1)(n+1)+|A^*|}.$$

Proof: Let $S_v\in F_v(G\circ H)$. For $v\in V(H)$, let A_v denote a local binding set of H and note that $|N(A^*)|/|A^*|\leqslant|N(A_v)|/|A_v|$, i.e., $\mathrm{bind}(H)\leqslant\mathrm{bind}_v(H)$. Assume $|A^*|=a$.

If $v\in V(G)$ or $v\in A^*$ then

$$|S_v|=(m-1)n+m-1+|A^*|=(m-1)(n+1)+|A^*|,$$
$$|N(S_v)|=(m-1)(n+1)+|N(A^*)|+1.$$

If $v\in V(H)$ and $v\notin A^*$ then

$$|S_v|=(m-1)n+m-1+|A_v|=(m-1)(n+1)+|A_v|$$
$$|N(S_v)|=(m-1)(n+1)+|N(A_v)|+1.$$

It is easy to see that

$$\frac{(m-1)(n+1)+|N(A^*)|+1}{(m-1)(n+1)+|A^*|}\leqslant\frac{(m-1)(n+1)+|N(A_v)|+1}{(m-1)(n+1)+|A_v|}.$$

So we have

$$\sum_{v\in V(G\circ H)} \mathrm{bind}_v(G\circ H)$$
$$= ((m-1)(n+1)+a)\frac{(m-1)(n+1)+|N(A^*)|+1}{(m-1)(n+1)+|A^*|} + (n-a)\frac{(m-1)(n+1)+|N(A_v)|+1}{(m-1)(n+1)+|A_v|}$$
$$\geqslant (mn+m)\times\frac{(m-1)(n+1)+|N(A^*)|+1}{(m-1)(n+1)+|A^*|}.$$

Thus

$$\mathrm{bind}_{\mathrm{av}}(G\circ H) = \frac{1}{mn+m}\sum_{v\in V(G\circ H)}\mathrm{bind}_v(G\circ H) \geqslant \frac{(m-1)(n+1)+|N(A^*)|+1}{(m-1)(n+1)+|A^*|}.$$

□

The lower bound in Theorem 14 is the best possible for some graphs as in Corollary 3 and 4.

Corollary 3 *Let G be a connected graph of order m. Then*

$$\mathrm{bind}_{\mathrm{av}}(G\circ K_n) = \frac{(m-1)(n+1)+n}{(m-1)(n+1)+1}.$$

Corollary 4 *Let G be a connected graph of order m. If n is even, then*

$$\mathrm{bind}_{\mathrm{av}}(G\circ P_n) = \frac{(m-1)(n+1)+\frac{n}{2}+1}{(m-1)(n+1)+\frac{n}{2}},$$
$$\mathrm{bind}_{\mathrm{av}}(G\circ C_n) = \frac{(m-1)(n+1)+\frac{n}{2}+1}{(m-1)(n+1)+\frac{n}{2}}.$$

Cartesian product of graphs

Definition 4 [Ref. 5] The Cartesian product $G_1\times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1)\times V(G_2)$, where (u_1,u_2) is adjacent to (v_1,v_2) if either $u_1=v_1$ and u_2 is adjacent to v_2 or $u_2=v_2$ and u_1 is adjacent to v_1.

Theorem 15 (Ref. 11) *For graphs G and H, if* $\mathrm{bind}(G)\geqslant 1$*, then* $\mathrm{bind}(G\times H)\geqslant 1$.

Theorem 16 *For graphs G and H, if* $\mathrm{bind}(G)\geqslant 1$*, then* $\mathrm{bind}_{\mathrm{av}}(G\times H)\geqslant 1$.

Proof: By Theorem 15, if $\mathrm{bind}(G)\geqslant 1$, then $bind(G\times H)\geqslant 1$. It follows from Theorem 5 that $\mathrm{bind}_{\mathrm{av}}(G\times H)\geqslant \mathrm{bind}(G\times H)\geqslant 1$. □

The lower bound in Theorem 16 is the best possible for come graphs as in Corollary 5.

Corollary 5 *Let m and n be even integers. Then*
(i) $\mathrm{bind}_{\mathrm{av}}(P_m\times P_n)=1$.
(ii) $\mathrm{bind}_{\mathrm{av}}(P_m\times C_n)=1$.
(iii) $\mathrm{bind}_{\mathrm{av}}(C_m\times C_n)=1$.

CONCLUSIONS

In this study, a new graph theoretical parameter, namely, the average binding number, as the average of the local binding number of every vertex of a graph, has been presented for the network vulnerability. Additionally, the stability of popular interconnection networks has been studied and the average binding numbers have been computed. The average binding number gives the meaning that, if $\mathrm{bind}_{\mathrm{av}}(G)$ is large, then the vertices of G are well bound together in the sense that G has a lot of fairly well distributed edges.

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