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Matrix inequalities for unitarily invariant norms

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ABSTRACT: In this study, we present some matrix inequalities for unitarily invariant norms. Firstly, we present an inequality for unitarily invariant norms. As a consequence of this result, Huang-Peng-Zou's result follows immediately. Furthermore, we also establish inequalities for weak log-majorizations and unitarily invariant norms related to question of Bourin's.

KEYWORDS: t-geometric mean, positive semidefinite matrices, weak log-majorization

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INTRODUCTION

Let $\mathscr{M}_n$ be the space of $n\times n$ complex matrices. Let $\lambda_j(A)$, $j=1,2,\dots,n$, be the eigenvalues of $A\in\mathscr{M}_n$ repeated according to multiplicity, and $|\lambda(A)| := (|\lambda_1(A)|, |\lambda_2(A)|, \dots, |\lambda_n(A)|)$ with $|\lambda_1(A)| \geqslant |\lambda_2(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$. For $A\in\mathscr{M}_n$, the singular values of A is denoted by $\sigma_j(A)$, $j=1,2,\dots,n$, i.e., the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity, where A^* is the conjugate transpose of A. Let $\sigma(A) := (\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A))$ be the vector of the singular values of A. For two Hermitian matrices $A, B\in\mathscr{M}_n$, $A\leqslant (<) B$ means $B-A$ is a positive semidefinite (definite) matrix. A norm $\|\cdot\|$ on $\mathscr{M}_n$ is called a unitarily invariant norm if $\|UAV\|=\|A\|$ for A, U, $V\in\mathscr{M}_n$ with U, V are unitary matrices.

Let $\Phi(\cdot)$ be the corresponding symmetric gauge function of the unitarily invariant norms $\|\cdot\|$. Then $\|A\|=\Phi(\{\sigma_i(A)\}_{i=1}^n)$ for all $A\in\mathscr{M}_n$. Examples in this class are the Schatten p-norms and Ky Fan k-norms. I_n is the identity matrix of $\mathscr{M}_n$. The usual operator norm denoted by $\|\cdot\|_\infty$ is $\|A\|_\infty=\sigma_1(A)$ for $A\in\mathscr{M}_n$.

Let us recall some definitions of majorization. Given a real vector $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, we rearrange its components as $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$. For $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$, if

$$\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]},\ k=1,2,\dots,n,$$

then we say that x is weakly majorized by y and denotes by $x\prec_{\mathrm{w}} y$. If $x\prec_{\mathrm{w}} y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then we say that x is majorized by y and denotes by $x\prec y$. Further, if $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n)\in\mathbb{R}_+^n$ and

$$\prod_{i=1}^k x_{[i]} \leqslant \prod_{i=1}^k y_{[i]},\ k=1,2,\dots,n,$$

then we say that x is weakly log-majorized by y and denotes by $x\prec_{w\log} y$. If $x\prec_{w\log} y$ and $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$, then we say that x is log-majorized by y and denotes by $x\prec_{\log} y$. It is well-known that if $x\prec_{\mathrm{w}\log} y$, then $x\prec_{\mathrm{w}} y$.

Let $A\in\mathscr{M}_n$ be a Hermitian matrix with eigenvalues $\lambda_j(A)$ $(j=1,2,\dots,n)$. Then the spectral theorem states: there is a diagonal matrix $\Lambda=\mathrm{diag}(\lambda_1(A),\lambda_2(A),\dots,\lambda_n(A))$ such that

$$A=U\Lambda U^*,$$

where U is a unitary matrix.

Let f be a real continuous function on an interval $[a,b]$, if $A\in\mathscr{M}_n$ is a Hermitian matrix with eigenvalues $\lambda_j(A)\in[a,b]$, $j=1,2,\dots,n$. Then by the spectral theorem of A, $f(A)$ is defined by

$$f(A)=U\Lambda_f U^*,$$

where U is a unitary matrix and $\Lambda_f=\mathrm{diag}(f(\lambda_1(A)), f(\lambda_2(A)),\dots,f(\lambda_n(A)))$.

Bhatia et al[1] proved that; if $A,B\in\mathscr{M}_n$ be two positive semidefinite matrices, then

$$\|A^m+B^m\|\leqslant\|(A+B)^m\| \tag{1}$$

holds for any positive integer m and any unitarily invariant norm $\|\cdot\|$. Ando et al[2] generalized inequality

(1) that; if $A, B \in \mathscr{M}_n$ be two positive semidefinite matrices, then

$$\|A^p + B^p\| \leqslant \|(A+B)^p\| \tag{2}$$

holds for any positive real number p with $1 \leqslant p < \infty$ and any unitarily invariant norm $\|\cdot\|$. Bourin et al[3] obtained a more generalization of inequality (2). They presented; if $A, B \in \mathscr{M}_n$ be two positive semidefinite matrices and $f : [0, \infty) \to [0, \infty)$ be a convex function with $f(0) = 0$, then

$$\|f(A) + f(B)\| \leqslant \|f(A+B)\| \tag{3}$$

holds for any unitarily invariant norm $\|\cdot\|$. Recently, Huang et al[4] obtained; if $A, B \in \mathscr{M}_n$ and suppose that p, q be real numbers with $p > 1$ and $1/p + 1/q = 1$, then

$$\left\|A|A|^{m-1} + B|B|^{m-1}\right\| \leqslant \left\|\left(|A|^m + |B|^m\right)^{p/2}\right\|^{1/p} \cdot \left\|\left(|A^*|^m + |B^*|^m\right)^{q/2}\right\|^{1/q} \tag{4}$$

holds for any positive integer m and for any unitarily invariant norm $\|\cdot\|$. If $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B, respectively. Then inequality (4) can be rewritten as

$$\|U|A|^m + V|B|^m\| \leqslant \left\|\left(|A|^m + |B|^m\right)^{p/2}\right\|^{1/p} \cdot \left\|\left(|A^*|^m + |B^*|^m\right)^{q/2}\right\|^{1/q}. \tag{5}$$

On the other hand, Hayajneh et al[5] and Liu et al[6] were independently obtained: If $A_i, B_i \in \mathscr{M}_n$ be positive semidefinite matrices with $A_iB_i = B_iA_i$, $i = 1, 2, \dots, m$, then for all unitarily invariant norms $\|\cdot\|$,

$$\left\|\left(\sum_{i=1}^m A_i^{1/2} B_i^{1/2}\right)^2\right\| \leqslant \left\|\left(\sum_{i=1}^m A_i\right)^{1/2}\left(\sum_{i=1}^m B_i\right)\left(\sum_{i=1}^m A_i\right)^{1/2}\right\|. \tag{6}$$

Inequality (6) is a refinement of the following inequality obtained by Audenaert[7]: If $A_i, B_i \in \mathscr{M}_n$ be positive semidefinite matrices with $A_iB_i = B_iA_i$, $i = 1, 2, \dots, m$, then for all unitarily invariant norms $\|\cdot\|$,

$$\left\|\left(\sum_{i=1}^m A_i^{1/2} B_i^{1/2}\right)^2\right\| \leqslant \left\|\left(\sum_{i=1}^m A_i\right)\left(\sum_{i=1}^m B_i\right)\right\|. \tag{7}$$

Hoa[8] and Lin[9] presented different proofs for inequality (7), respectively. Inequality (7) gave an affirmative answer to Bourin's question. Given two positive semidefinite matrices $A, B \in \mathscr{M}_n$ and two positive real numbers p, q, is it true that

$$\left\|A^{p+q} + B^{p+q}\right\| \leqslant \|(A^p + B^p)(A^q + B^q)\|? \tag{8}$$

Refinements and improvements for unitarily invariant norms have been extensively studied. Many researchers, e.g., Bhatia[10], Fujii[11], Hu[12,13], Kapil[14], Kittaneh[15], Kuzma[16], Matharu[17], paid attention to the improvement and generalization of inequalities for unitarily invariant norms.

In this study, we present some matrix inequalities for unitarily invariant norms. We present a generalization of inequality (5) and establish inequalities for weak log-majorizations and unitarily invariant norms related to Bourin's question.

MATRIX INEQUALITIES FOR UNITARILY INVARIANT NORMS

This section mainly presents a generalization of inequality (5) for unitarily invariant norms and inequalities for weak log-majorizations and unitarily invariant norms related to Bourin's question. To achieve the goal, we require lemmas presented in Ref. 18.

Lemma 1 *If $A, B \in \mathscr{M}_n$ with $A, B \geqslant 0$, then the matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite if and only if $X = A^{1/2}KB^{1/2}$ for some contraction K, i.e., $K^*K \leqslant I_n$.*

Lemma 2 *$A \geqslant 0$ if and only if $\begin{bmatrix} A & A \\ A & A \end{bmatrix} \geqslant 0$.*

The next lemma was obtained by Horn[19].

Lemma 3 *If $A, B \in \mathscr{M}_n$, then*

$$\sigma(AB) \prec_{\log} \{\sigma_i(A)\sigma_i(B)\}_{i=1}^n.$$

The Lemma 4 was obtained by Matharu et al[17].

Lemma 4 *If $A, B \in \mathscr{M}_n$ with A, $B > 0$ and $t \in [0,1]$, then*

$$\lambda(A \sharp_t B) \prec_{w\log} \lambda(A^{1-t}B^t),$$

where $A \sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is the t-geometric mean of A and B [20].

The next lemma was given by Hiai[21].

Lemma 5 *If $A, B \in \mathscr{M}_n$ with A, $B > 0$ and $t \in [0,1]$, then*

$$\lambda\left(\left(A^{1/2}BA^{1/2}\right)^r\right) \prec_{w\log} \lambda\left(A^{r/2}B^rA^{r/2}\right),$$

for $r \geqslant 1$.

The next Lemma 6 is the famous Weyl's theorem[19] on the singular values and the eigenvalues of a matrix.

Lemma 6 *If $A\in\mathscr{M}_n$, then*

$$|\lambda(A)| \prec_{\log} \sigma(A).$$

In the following, we present the famous Fan dominance theorem[19].

Lemma 7 *If $A,B\in\mathscr{M}_n$, then*

$$\sigma(A) \prec_{\mathrm{w}} \sigma(B) \iff \|A\| \leqslant \|B\|$$

for any unitarily invariant norm $\|\cdot\|$.

It is now time to present the following theorem.

Theorem 1 *If $A,B\in\mathscr{M}_n$ and $f:[0,\infty)\to[0,\infty)$ is a continuous function with $f(0)=0$, then*

$$\|Uf(|A|)+Vf(|B|)\| \leqslant \left\|\left(f(|A|)+f(|B|)\right)^{p/2}\right\|^{1/p} \cdot \left\|\left(f(|A^*|)+f(|B^*|)\right)^{q/2}\right\|^{1/q} \qquad (9)$$

holds for positive real numbers p, q with $1/p+1/q=1$ and any unitarily invariant norm $\|\cdot\|$, where U and V are unitary matrices with $A=U|A|$ and $B=V|B|$, respectively.

Proof: Let $A=U|A|$ and $B=V|B|$ be the polar decompositions of A and B, respectively. By Lemma 2, we have

$$\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} A & |A| \\ |A| & |A| \end{bmatrix}\begin{bmatrix} I_n & 0 \\ 0 & U^* \end{bmatrix} \geqslant 0. \qquad (10)$$

Putting

$$W=\frac{1}{\sqrt{2}}\begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix},$$

then W is a unitary matrix and

$$W\begin{bmatrix} |A| & 0 \\ 0 & 0 \end{bmatrix}W^* = \tfrac{1}{2}\begin{bmatrix} |A| & |A| \\ |A| & |A| \end{bmatrix}. \qquad (11)$$

Combining equalities (10) with (11), we obtain

$$\tfrac{1}{2}\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & U \end{bmatrix} W \begin{bmatrix} |A| & 0 \\ 0 & 0 \end{bmatrix} W^* \begin{bmatrix} I_n & 0 \\ 0 & U^* \end{bmatrix}. \qquad (12)$$

Since f is a nonnegative function on $[0,\infty)$ with $f(0)=0$, by equality (12) we obtain

$$f\left(\tfrac{1}{2}\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix} f(|A|) & f(|A|)U^* \\ Uf(|A|) & f(|A^*|) \end{bmatrix}. \qquad (13)$$

Similarly, we also have

$$f\left(\tfrac{1}{2}\begin{bmatrix} |B| & B^* \\ B & |B^*| \end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix} f(|B|) & f(|B|)V^* \\ Vf(|B|) & f(|B^*|) \end{bmatrix}. \qquad (14)$$

It follows from equalities (13) and (14) that

$$\begin{bmatrix} f(|A|)+f(|B|) & f(|A|)U^*+f(|B|)V^* \\ Uf(|A|)+Vf(|B|) & f(|A^*|)+f(|B^*|) \end{bmatrix} = 2f\left(\tfrac{1}{2}\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}\right) + 2f\left(\tfrac{1}{2}\begin{bmatrix} |B| & B^* \\ B & |B^*| \end{bmatrix}\right) \geqslant 0. \qquad (15)$$

By inequality (15) and Lemma 1, there is a contraction K such that

$$f(|A|)U^*+f(|B|)V^* = \left(f(|A|)+f(|B|)\right)^{1/2}K\left(f(|A^*|)+f(|B^*|)\right)^{1/2},$$

or equivalently,

$$Uf(|A|)+Vf(|B|) = \left(f(|A^*|)+f(|B^*|)\right)^{1/2}K^*\left(f(|A|)+f(|B|)\right)^{1/2}. \qquad (16)$$

According to (16), we have

$$\begin{aligned} &\prod_{j=1}^{k}\sigma_j\left(Uf(|A|)+Vf(|B|)\right) \\ &\leqslant \prod_{j=1}^{k}\sigma_j\left(\left(f(|A^*|)+f(|B^*|)\right)^{\frac{1}{2}}\right)\sigma_j\left(K^*\left(f(|A|)+f(|B|)\right)^{\frac{1}{2}}\right) \\ &\leqslant \prod_{j=1}^{k}\sigma_j\left(\left(f(|A^*|)+f(|B^*|)\right)^{\frac{1}{2}}\right)\sigma_j\left(\left(f(|A|)+f(|B|)\right)^{\frac{1}{2}}\right), \end{aligned} \qquad (17)$$

for $k=1,2,\dots,n$, the first inequality is due to Lemma 3, and the second by the contractive of K^*.

Since weak log-majorization implies weak majorization, we obtain the following weak majorization from (17).

$$\sigma\left(Uf(|A|)+Vf(|B|)\right) \prec_{\mathrm{w}} \left\{\sigma_j\left(\left(f(|A^*|)+f(|B^*|)\right)^{\frac{1}{2}}\right)\sigma_j\left(\left(f(|A|)+f(|B|)\right)^{\frac{1}{2}}\right)\right\}_{j=1}^{n}. \qquad (18)$$

Let $\Phi(\cdot)$ be the corresponding symmetric gauge function for the unitarily invariant norm $\|\cdot\|$. By the Cauchy-Schwarz inequality for $\Phi(\cdot)$ we have[22]

$$\begin{aligned} &\Phi\left(\left\{\sigma_j\left(\left(f(|A^*|)+f(|B^*|)\right)^{\frac{1}{2}}\right)\sigma_j\left(\left(f(|A|)+f(|B|)\right)^{\frac{1}{2}}\right)\right\}_{j=1}^{n}\right) \\ &\leqslant \Phi\left(\sigma\left(\left(f(|A^*|)+f(|B^*|)\right)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}}\Phi\left(\sigma\left(\left(f(|A|)+f(|B|)\right)^{\frac{q}{2}}\right)\right)^{\frac{1}{q}} \end{aligned} \qquad (19)$$

for positive real numbers p, q with $1/p+1/q=1$. On the other hand, by inequality (18), we obtain

$$\Phi(\sigma(Uf(|A|)+Vf(|B|))) \leqslant \Phi\left(\left\{\sigma_j\big((f(|A^*|)+f(|B^*|))^{\frac{1}{2}}\big)\sigma_j\big((f(|A|)+f(|B|))^{\frac{1}{2}}\big)\right\}_{j=1}^{n}\right). \tag{20}$$

Noting that $\Phi(\sigma(A))=\|A\|$ for $A\in\mathscr{M}_n$ and (19) and (20) imply the desired result (9).

$$\|Uf(|A|)+Vf(|B|)\| \leqslant \left\|(f(|A|)+f(|B|))^{\frac{p}{2}}\right\|^{\frac{1}{p}} \left\|(f(|A^*|)+f(|B^*|))^{\frac{q}{2}}\right\|^{\frac{1}{q}}.$$

□

Remark 1 Putting $f(x)=x^m$ for some positive integer number m, (5) follows immediately from (10).

Remark 2 Let f be a nonnegative convex function on $[0,\infty)$ with $f(0)=0$ and $p=q=2$. Then by inequality (10) we have

$$\|Uf(|A|)+Vf(|B|)\| \leqslant \|f(|A|)+f(|B|)\|^{\frac{1}{2}}\,\|f(|A^*|)+f(|B^*|)\|^{\frac{1}{2}}, \tag{21}$$

and by inequalities (3) and (21), the following inequality holds

$$\|Uf(|A|)+Vf(|B|)\| \leqslant \|f(|A|+|B|)\|^{\frac{1}{2}}\,\|f(|A^*|+|B^*|)\|^{\frac{1}{2}}. \tag{22}$$

If $A,B\geqslant 0$, then $U=V=I_n$. Hence, inequality (3) follows immediately from (22).

Next, we give weak log-majorization inequalities for positive semidefinite matrices.

Theorem 2 *If $A,B\in\mathscr{M}_n$ with A, $B>0$ and $t\in[0,1]$, then*

$$\lambda\big((A\sharp_t B)^r\big) \prec_{w\log} \lambda\big((B^{rts/2}A^{(1-t)rs}B^{rts/2})^{1/s}\big) \prec_{w\log} \lambda\big(\big|A^{(1-t)rs}B^{rts}\big|^{1/s}\big), \tag{23}$$

holds for positive r, s with $rs\geqslant 1$.

Proof: By Lemma 4 we have, for $k=1,2,\dots,n$,

$$\prod_{j=1}^{k}\lambda_j(A\sharp_t B) \leqslant \prod_{j=1}^{k}\lambda_j(A^{1-t}B^t) = \prod_{j=1}^{k}\lambda_j(B^tA^{1-t}) = \prod_{j=1}^{k}\lambda_j\Big(B^{t/2}A^{1-t}B^{t/2}\Big). \tag{24}$$

By Lemma 5 we obtain, for $rs\geqslant 1$,

$$\prod_{j=1}^{k}\lambda_j^{rs}\Big(B^{t/2}A^{1-t}B^{t/2}\Big) = \prod_{j=1}^{k}\lambda_j\big((B^{t/2}A^{1-t}B^{t/2})^{rs}\big) \leqslant \prod_{j=1}^{k}\lambda_j\Big(B^{rst/2}A^{(1-t)rs}B^{rst/2}\Big). \tag{25}$$

Combining inequalities (24) and (25), we obtain for $k=1,2,\dots,n$ and $rs\geqslant 1$,

$$\prod_{j=1}^{k}\lambda_j^{r}(A\sharp_t B) \leqslant \prod_{j=1}^{k}\lambda_j\big((B^{rst/2}A^{(1-t)rs}B^{rst/2})^{1/s}\big). \tag{26}$$

On the other hand, by Lemma 6, we have

$$\prod_{j=1}^{k}\lambda_j\big(B^{rst/2}A^{(1-t)rs}B^{rst/2}\big) = \prod_{j=1}^{k}\lambda_j\big(A^{(1-t)rs}B^{rts}\big) \leqslant \prod_{j=1}^{k}\sigma_j\big(A^{(1-t)rs}B^{rst}\big),$$

or equivalently,

$$\prod_{j=1}^{k}\lambda_j\big((B^{\frac{rst}{2}}A^{(1-t)rs}B^{\frac{rst}{2}})^{\frac{1}{s}}\big) = \prod_{j=1}^{k}\lambda_j^{\frac{1}{s}}\big(A^{(1-t)rs}B^{rst}\big) \leqslant \prod_{j=1}^{k}\sigma_j^{\frac{1}{s}}\big(A^{(1-t)rs}B^{rst}\big) = \prod_{j=1}^{k}\lambda_j\big(\big|A^{(1-t)rs}B^{rst}\big|^{\frac{1}{s}}\big). \tag{27}$$

Thus the desired inequality (23) follows from (26) and (27). □

Since weak log-majorization implies weak majorization, Theorem 2 and Lemma 7 imply the following theorem:

Theorem 3 *If $A,B\in\mathscr{M}_n$ with A, $B>0$ and $t\in[0,1]$, then for all unitarily invariant norms $\|\cdot\|$ on $\mathscr{M}_n$,*

$$\|(A\sharp_t B)^r\| \leqslant \left\|\Big(B^{rst/2}A^{(1-t)rs}B^{rst}\Big)^{1/s}\right\| \leqslant \left\|\big|A^{(1-t)rs}B^{rst}\big|^{1/s}\right\| \tag{28}$$

holds for positive r, s with $rs\geqslant 1$.

Remark 3 Hoa obtained the following result[8]; if $A,B\in\mathscr{M}_n$ with A, $B>0$ and $t\in[0,1]$, then for all unitarily invariant norms $\|\cdot\|$ on $\mathscr{M}_n$,

$$\left\|(A\sharp_t B)^r\right\| \leqslant \left\|\left(B^{rst/2}A^{(1-t)rs}B^{rst}\right)^{1/s}\right\| \leqslant \left\|\left|A^{(1-t)rs}B^{rst}\right|^{1/s}\right\| \tag{29}$$

holds for $r \geqslant 1$, $s > 0$. Since $\{(s,r) \mid 0 < r < 1 < rs\} \cap \{(s,r) \mid r \geqslant 1, s > 0\} = \varnothing$, (28) is a complement of (29).

Theorem 4 *Let $A_i, B_i \in \mathscr{M}_n$ be positive definite matrices, $i = 1,2,\dots,m$. Then*

$$\lambda\Bigl(\Bigl(\sum_{i=1}^m A_i\sharp_t B_i\Bigr)^r\Bigr) \prec_{w\log} \lambda\Bigl(\Bigl(\Bigl(\sum_{i=1}^m B_i\Bigr)^{\frac{rst}{2}}\Bigl(\sum_{i=1}^m A_i\Bigr)^{rst}\Bigl(\sum_{i=1}^m B_i\Bigr)^{\frac{rst}{2}}\Bigr)^{\frac{1}{s}}\Bigr) \prec_{w\log} \lambda\Bigl(\Bigl|\Bigl(\sum_{i=1}^m A_i\Bigr)^{(1-t)rs}\Bigl(\sum_{i=1}^m B_i\Bigr)^{rst}\Bigr|^{\frac{1}{s}}\Bigr), \tag{30}$$

where for positive r, s with $rs \geqslant 1$ and $t \in [0,1]$.

Proof: By the monotonicity of the operator mean $\sharp_t$ for $t \in [0,1]$, we have

$$\sum_{i=1}^m A_i\sharp_t B_i \leqslant \Bigl(\sum_{i=1}^m A_i\Bigr)\sharp_t\Bigl(\sum_{i=1}^m B_i\Bigr),$$

which implies

$$\lambda_j\Bigl(\sum_{i=1}^m A_i\sharp_t B_i\Bigr) \leqslant \lambda_j\Bigl(\Bigl(\sum_{i=1}^m A_i\Bigr)\sharp_t\Bigl(\sum_{i=1}^m B_i\Bigr)\Bigr) \tag{31}$$

for $j = 1,2,\dots,n$. According to Theorem 2, we obtain

$$\lambda\Bigl(\Bigl(\Bigl(\sum_{i=1}^m A_i\Bigr)\sharp_t\Bigl(\sum_{i=1}^m B_i\Bigr)\Bigr)^r\Bigr) \prec_{w\log} \lambda\Bigl(\Bigl(\Bigl(\sum_{i=1}^m B_i\Bigr)^{\frac{rst}{2}}\Bigl(\sum_{i=1}^m A_i\Bigr)^{rst}\Bigl(\sum_{i=1}^m B_i\Bigr)^{\frac{rst}{2}}\Bigr)^{\frac{1}{s}}\Bigr) \prec_{w\log} \lambda\Bigl(\Bigl|\Bigl(\sum_{i=1}^m A_i\Bigr)^{(1-t)rs}\Bigl(\sum_{i=1}^m B_i\Bigr)^{rst}\Bigr|^{\frac{1}{s}}\Bigr). \tag{32}$$

Combining inequalities (31) and (32), we obtain the desired inequality (30). □

Since weak log-majorization implies weak majorization, by Theorem 4 and Lemma 7, we obtain the following theorem.

Theorem 5 *If $A_i, B_i \in \mathscr{M}_n$ with A_i, $B_i > 0$, $i = 1,2,\dots,m$, then*

$$\Bigl\|\Bigl(\sum_{i=1}^m A_i\sharp_t B_i\Bigr)^r\Bigr\| \leqslant \Bigl\|\Bigl(\Bigl(\sum_{i=1}^m B_i\Bigr)^{\frac{rst}{2}}\Bigl(\sum_{i=1}^m A_i\Bigr)^{(1-t)rs}\Bigl(\sum_{i=1}^m B_i\Bigr)^{\frac{rst}{2}}\Bigr)^{\frac{1}{s}}\Bigr\| \leqslant \Bigl\|\Bigl|\Bigl(\sum_{i=1}^m A_i\Bigr)^{(1-t)rs}\Bigl(\sum_{i=1}^m B_i\Bigr)^{rst}\Bigr|^{\frac{1}{s}}\Bigr\|$$

holds for any unitarily invariant norm $\|\cdot\|$, positive r, s with $rs \geqslant 1$, and $t \in [0,1]$.

Taking $r = 2$ and $s = 1$ in Theorem 5, we have the following corollary.

Corollary 1 *If $A_i, B_i \in \mathscr{M}_n$ with A_i, $B_i > 0$, $i = 1,2,\dots,m$, then*

$$\Bigl\|\Bigl(\sum_{i=1}^m A_i\sharp_t B_i\Bigr)^2\Bigr\| \leqslant \Bigl\|\Bigl(\sum_{i=1}^m B_i\Bigr)^t\Bigl(\sum_{i=1}^m A_i\Bigr)^{2(1-t)}\Bigl(\sum_{i=1}^m B_i\Bigr)^t\Bigr\| \leqslant \Bigl\|\Bigl(\sum_{i=1}^m A_i\Bigr)^{2(1-t)}\Bigl(\sum_{i=1}^m B_i\Bigr)^{2t}\Bigr\| \tag{33}$$

holds for any unitarily invariant norm and $t \in [0,1]$.

Since $A\sharp_t B = A^{1-t}B^t$ when $AB = BA$ for A, $B > 0$, the following remark holds.

Remark 4 Let $A_i, B_i \in \mathscr{M}_n$ be positive definite matrices with $A_iB_i = B_iA_i$, $i = 1,2,\dots,m$. From (3), by taking $f(x) = x^2$, we have

$$\Bigl\|\sum_{i=1}^m (A_i^{1-t}B_i^t)^2\Bigr\| \leqslant \Bigl\|\Bigl(\sum_{i=1}^m A_i^{1-t}B_i^t\Bigr)^2\Bigr\|. \tag{34}$$

Combining inequalities (33) and (34), we obtain

$$\Bigl\|\sum_{i=1}^m (A_i^{1-t}B_i^t)^2\Bigr\| \leqslant \Bigl\|\Bigl(\sum_{i=1}^m B_i\Bigr)^t\Bigl(\sum_{i=1}^m A_i\Bigr)^{2(1-t)}\Bigl(\sum_{i=1}^m B_i\Bigr)^t\Bigr\| \leqslant \Bigl\|\Bigl(\sum_{i=1}^m A_i\Bigr)^{2(1-t)}\Bigl(\sum_{i=1}^m B_i\Bigr)^{2t}\Bigr\|. \tag{35}$$

Inequality (35) is a generalization of inequalities (6) and (7). On the other hand, let A and B be two positive definite matrices and p, q be two positive real numbers. Taking $m=2$, $t=\frac{1}{2}, A_1=A^p, B_1=A^q$, $A_2=B^p$, $B_2=B^q$ in inequality (35), we have

$$\left\|A^{p+q}+B^{p+q}\right\| \leqslant \left\|(A^p+B^p)^{\frac{1}{2}}(A^q+B^q)(A^p+B^p)^{\frac{1}{2}}\right\| \leqslant \|(A^p+B^p)(A^q+B^q)\|,$$

which is (8). This gives an affirmative answer to Bourin's question.

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