

Exponential stability of travelling waves for a general reaction-diffusion equation with spatio-temporal delays

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ABSTRACT: This paper is concerned with the exponential stability of travelling waves for a general reaction-diffusion equation with spatio-temporal delays. Here the travelling waves may be monotone or non-monotone. More precisely, by means of the weighted-energy method and the nonlinear Halanay inequality, we can prove that the travelling waves of this equation are exponentially stable, when the initial perturbation around the wave is small in a weighted norm. In the end, we apply our results to some models.

KEYWORDS: delayed reaction-diffusion equation, weighted energy method, Halanay inequality

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INTRODUCTION

In this paper, we will study a general reaction-diffusion equation with spatio-temporal delays

$$u_t(t, x) = du_{xx}(t, x) + F(u(t, x), \int_{-\infty}^{\infty} J(x-y)f(u(t-r, y))dy), \quad (1)$$

$t \geq 0, x \in \mathbb{R}$, combined with the initial datum

$$u(s, x) = u_0(s, x), \quad s \in [-r, 0], x \in \mathbb{R}, \quad (2)$$

where f may not be monotone or quasi-monotone, $d > 0$ and $r \geq 0$. Firstly, we give following assumptions for (1).

(A₁) $J(y) \geq 0, J(y) = J(-y), \int_{-\infty}^{\infty} J(y)dy = 1,$ and $\int_{-\infty}^{\infty} J(y)e^{\lambda y}dy < \infty$ for $\lambda > 0$.

(A₂) For $K > 0, f \in C^2([0, K], [0, f(K)]), f(0) = 0, f'(0) > 0, 0 < f(x) \leq f(K)$ for $x \in (0, K]$, and $|f(x) - f(y)| \leq f'(0)|x - y|$ for $x, y \in [0, K]$.

(A₃) $F \in C^2([0, K] \times [0, f(K)], \mathbb{R}), F(0, 0) = 0, F(K, f(K)) = 0, F(x, f(x)) > 0, \partial_1 F(x, y) \leq 0, \partial_2 F(x, y) > 0, F(x, y) \leq \partial_1 F(0, 0)x + \partial_2 F(0, 0)y,$ and $\partial_i F(x, y) \leq \partial_i F(0, 0)$ for $(x, y) \in [0, K] \times [0, f(K)]$ when $i = 1, 2,$ and $\partial_1 F(x, y) = \frac{\partial F}{\partial x}(x, y), \partial_2 F(x, y) = \frac{\partial F}{\partial y}(x, y).$

(A₄) There exist $0 < \theta < K$ and $0 < \eta < K$ such that f is increasing on $[0, \theta]$ and $F(x, y) = 0$ has a solution $x \in (0, \theta)$ for each $y \in (0, \eta).$

(A₅) $\partial_2 F(K, f(K))f'(K) + \partial_1 F(K, f(K)) \leq 0$ for any $r > 0,$ or $\partial_2 F(K, f(K))f'(K) + \partial_1 F(K, f(K)) > 0$ for $0 < r < \bar{r},$ when

$$\bar{r} = \frac{\pi - \arctan[-(\partial_1 F(K, f(K)))^{-1}M]}{M}, \quad (3)$$

$$M = \sqrt{(\partial_2 F(K, f(K))f'(K))^2 - (\partial_1 F(K, f(K)))^2}.$$

It is well known that reaction-diffusion equations with delays are often used to demonstrate practical phenomena. And travelling wave solutions of these equations have been studied universally because of their important applications. Throughout this paper, a travelling wave solution of (1) always refers to a pair $(\phi, c),$ where $\phi = \phi(\xi), \xi = x + ct,$ is a function on \mathbb{R} satisfying

$$c\phi'(\xi) = d\phi''(\xi) + F(\phi(\xi), \int_{-\infty}^{\infty} J(y)f(\phi(\xi - y - cr))dy).$$

Furthermore, if ϕ is monotone and bounded, we call it a travelling wavefront.

For reaction-diffusion equations, monotonicity and quasi-monotonicity of reaction terms are common hypotheses to ensure the existence of travelling wave solutions. Without these hypotheses, many authors have studied the existence of travelling wave solutions. For example, Ma¹ established

the existence of travelling wave solutions for a class of delayed nonlocal equations without quasi-monotonicity by constructing two associated auxiliary functions and applying Schauder’s fixed point theorem. Similar results can be obtained^{2,3}, i.e., they proved the existence of non-monotone travelling wave solutions. Wang⁴ proved the existence of travelling wave solutions for (1) by using upper-lower solutions for associated integral equations and Schauder’s fixed point theorem. Then for a class of non-monotone reaction-diffusion equation with spatiotemporal delays, Xu et al⁵ obtained the existence and uniqueness of travelling wave solutions by presenting a new method to construct a closed and convex set and using Schauder’s fixed point theorem.

Besides the existence of travelling wave solutions of delayed reaction-diffusion equations, the stability of travelling waves has also attracted great attention. Let us provide some background on this topic in the literature. The first conclusion on the linearized stability of travelling wave solutions was established by Schaaf⁶ via the spectrum analysis method. Mei et al⁷ proved the nonlinear stability of the travelling wavefronts for the local Nicholson’s blowflies equation with a discrete delay by using a technical weighted-energy method. Subsequently, many researchers used this method to obtain the stability of travelling wavefronts for a variety of monostable reaction-diffusion equations with delays^{8,9} and further improved the former results to obtain global stability of travelling wavefronts by using both the comparison principle and the technical weighted-energy method^{10,11}.

Since the monotonicity of both the equations and the travelling wave solutions are necessary for these results of stability, therefore we cannot use the above methods to establish the stability of non-monotone travelling wave solutions. Wu et al¹² firstly studied the asymptotic stability of the non-monotone travelling wave solutions of delayed reaction-diffusion equations with crossing-monostability by adopting two ideal weight functions. Subsequently, by using the technical weighted-energy method and the nonlinear Halanay’s inequality, Lin et al¹³ obtained exponential decay estimate of all non-critical travelling wave solutions including the oscillating waves under some assumptions for a time-delayed reaction-diffusion equation. Since there is no result about stability of travelling wave solutions for (1), the main purpose of this study is the stability of the travelling wave solutions for (1) with non-monotone reaction terms.

PRELIMINARIES AND MAIN RESULTS

In this paper, we assume that $C > 0$ represents a generic constant and $C_i > 0$ denotes a particular constant. We let I denote an interval, typically $I = \mathbb{R}$. $L^2(I)$ is the space of the square integrable functions on I and $H^k(I)(k \geq 0)$ is the Sobolev space of L^2 -functions $f(x)$ defined on I satisfying $f^{(i)}(x) \in L^2(I), i = 1, \dots, k$. $L^2_\omega(I)$ denotes the weighted L^2 -space with a weight function $\omega, \omega : \mathbb{R} \rightarrow (0, \infty)$ is a locally integrable function, with norm defined by $\|f\|_{L^2_\omega(I)} = (\int_I \omega(x)f^2(x)dx)^{1/2}$. $H^k_\omega(I)$ is the weighted Sobolev space with the norm $\|f\|_{H^k_\omega(I)} = (\sum_{i=0}^k \int_I \omega(x)|f^{(i)}(x)|^2 dx)^{1/2}$. Furthermore, we let $T > 0$ be a constant, \mathcal{B} be a Banach space, and denote by $C([0, T]; \mathcal{B})$ the space of the \mathcal{B} -valued continuous functions on $[0, T]$. The spaces of the \mathcal{B} -valued functions on $[0, \infty)$ can be defined similarly. Furthermore, the space $\mathcal{C}_{\text{unif}}[-r, T], 0 < T \leq \infty$, is defined by

$$\mathcal{C}_{\text{unif}}[-r, T] := \{v(t, x) \in C([-r, T] \times \mathbb{R})\}$$

such that $\lim_{x \rightarrow \infty} e^{\mu_2 t} v(t, x)$ exists uniformly for $t \in [-r, T]$ and $\mu_2 > 0$, a constant to be defined later, and

$$\lim_{x \rightarrow \infty} v_x(t, x) = 0, \lim_{x \rightarrow \infty} v_{xx}(t, x) = 0$$

uniformly with respect to $t \in [-r, T]$. The nonlinear Halanay inequality is given as follows.

Lemma 1 (Ref. 13) *Let $z(t)$ be the solution of the following linear delay differential equation*

$$z'(t) + k_1 z(t) - k_2 z(t - r) = \alpha f(z(t)) + \beta g(z(t - r)),$$

$$z(s) = z_0(s), \quad s \in [-r, 0],$$

where k_1, k_2, α, β are any given numbers, and $f(z)$ and $g(z)$ satisfy

$$|f(z)| \leq C|z|^m, |g(z)| \leq C|z|^n, m > 1, n > 1.$$

If $|k_2| \leq k_1$ for $r > 0$ or $|k_2| > k_1$ for $0 < r < \bar{r}$,

$$\bar{r} = \frac{\pi - \arctan(k_1^{-1} \sqrt{|k_2|^2 - k_1^2})}{\sqrt{|k_2|^2 - k_1^2}},$$

then if $\|z_0\|_{L^\infty(-r,0)} \ll 1$,

$$|z(t)| \leq C \|z_0\|_{L^\infty(-r,0)} e^{-\nu t}, \quad t > 0,$$

for some $0 < \nu = \nu(k_1, k_2, r) < k_1$.

Theorem 1 (existence of travelling waves)

Assume that $(A_1) - (A_4)$ hold, there exist a minimal wave speed $c_* > 0$ and a corresponding number $\lambda_*(c_*)$ satisfying $\mathcal{P}(\lambda_*, c_*) = 0$ and $\frac{\partial}{\partial \lambda} \mathcal{P}(\lambda_*, c_*) = 0$ where

$$\mathcal{P}(\lambda, c) = d\lambda^2 - c\lambda + \partial_1 F(0, 0) + \partial_2 F(0, 0) f'(0) \int_{-\infty}^{\infty} J(y) e^{-\lambda(y+cr)} dy$$

is the characteristic equation. For any $c \geq c_*$, then (1) admits a travelling wave solution $\phi(\xi)$, $\xi = x + ct$, such that $0 < \phi(\xi) \leq K$, $\phi(-\infty) = 0$ and

$$0 < \liminf_{\xi \rightarrow \infty} \phi(\xi) \leq \limsup_{\xi \rightarrow \infty} \phi(\xi) \leq K.$$

If $\partial_1 F(x, y) \leq 0$, then $\lim_{\xi \rightarrow \infty} \phi(\xi) = K$. In addition, if f is nondecreasing on $[0, K]$, then $\phi(\xi)$ is nondecreasing on \mathbb{R} . While for $0 < c < c_*$, there has no travelling wave solution. Furthermore, when $c > c_*$, $\mathcal{P}(\lambda, c) = 0$ has two distinct positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda_2(c)$ such that

$$\mathcal{P}(\lambda, c) < 0 \text{ for } \lambda_1 < \lambda < \lambda_2.$$

When $c = c_*$, it holds that

$$\mathcal{P}(\lambda_*, c_*) = 0 \text{ for } \lambda_1 = \lambda_* = \lambda_2.$$

For any $c > c_*$, we define a weight function as

$$\omega(x) = e^{-2\lambda_*(x-x_0)} \quad \text{with } x_0 \gg 1,$$

where $\lambda_* > 0$ is defined in Theorem 1 and x_0 will be defined later. It is obvious that $\omega(x) \rightarrow \infty$ as $x \rightarrow -\infty$ and $\omega(x) \rightarrow 0$ as $x \rightarrow \infty$. Then we state the main result of this paper as follows.

Theorem 2 (stability of travelling waves)

Assume that $(A_1) - (A_5)$ hold. For the given travelling wave $\phi(\xi)$ of (1) with the speed $c > \tilde{c}$, where \tilde{c} satisfies

$$\begin{aligned} \tilde{c}\lambda_* - \partial_2 F(0, 0) f'(0) \left(\int_{-\infty}^{\infty} J(y) e^{-2\lambda_*(y+\tilde{c}r)} dy \right)^{1/2} \\ = 2d\lambda_*^2 + \partial_1 F(0, 0), \end{aligned}$$

if $u_0(s, x) - \phi(x + cs)$ is in $C([-r, 0]; C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \cap L^2([-r, 0]; H_\omega^2(\mathbb{R}))$ and

$$v_{0, \infty} := \lim_{x \rightarrow \infty} [u_0(s, x) - \phi(x + cs)] \in C[-r, 0]$$

exists uniformly with $s \in [-r, 0]$, then there exist constants $\delta_0 > 0$ and $0 < \mu < \min\{\mu_1, \mu_2\}$, μ_1, μ_2 will be defined later, such that when

$$\begin{aligned} \max_{s \in [-r, 0]} \|(u_0 - \phi)(s)\|_C^2 + \|(u_0 - \phi)(0)\|_{H_\omega^2}^2 \\ + \int_{-r}^0 \|(u_0 - \phi)(s)\|_{H_\omega^2}^2 ds \leq \delta_0^2, \end{aligned}$$

the unique solution $u(t, x)$ of (1) and (2) exists globally and $u(t, x) - \phi(x + ct)$ is in $C([-r, \infty); C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \cap L^2([-r, \infty); H_\omega^2(\mathbb{R})) \cap \mathcal{C}_{\text{unif}}[-r, \infty)$ with

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq \tilde{C} e^{-\mu t}, \quad t \geq 0,$$

where $\tilde{C} = \sqrt{C_{18}} \delta_0$, C_{18} will be determined later.

Remark 1 By Hölder inequality,

$$\int_{-\infty}^{\infty} J(y) e^{-\lambda_* y} dy \leq \left(\int_{-\infty}^{\infty} J(y) e^{-2\lambda_* y} dy \right)^{1/2}.$$

Then

$$\begin{aligned} c_* \lambda_* - \partial_2 F(0, 0) f'(0) \left(\int_{-\infty}^{\infty} J(y) e^{-2\lambda_*(y+c_*r)} dy \right)^{1/2} \\ - 2d\lambda_*^2 - \partial_1 F(0, 0) \leq 0, \end{aligned}$$

and we obtain that $\tilde{c} \geq c_*$.

A PRIORI ESTIMATE

For any $c > c_*$, let $\phi(\xi)$ be a travelling wave solution of (1) and $v(t, \xi) = u(t, x) - \phi(x + ct)$, $v_0(s, \xi) = u_0(s, x) - \phi(x + cs)$, where $\xi = x + ct$. Then it can be verified that $v(t, \xi)$ defined above satisfies

$$\begin{aligned} v_t(t, \xi) + cv_\xi(t, \xi) - dv_{\xi\xi}(t, \xi) - \partial_1 F(\xi)v(t, \xi) \\ = \partial_2 F(\xi) \int_{-\infty}^{\infty} J(y)[f(v + \phi) - f(\phi)] dy + Q, \end{aligned} \tag{4}$$

with $v(s, \xi) = v_0(s, \xi)$, $(s, \xi) \in [-r, 0] \times \mathbb{R}$, where $F(\xi) = F(\phi(\xi))$, $\int_{-\infty}^{\infty} J(y)f(\phi(\xi - y - cr)) dy$,

$$\begin{aligned} Q = F(v(t, \xi) + \phi(\xi)) \int_{-\infty}^{\infty} J(y)f(v + \phi) dy \\ - F(\xi) - \partial_1 F(\xi)v(t, \xi) \\ - \partial_2 F(\xi) \int_{-\infty}^{\infty} J(y)[f(v + \phi) - f(\phi)] dy. \end{aligned} \tag{5}$$

For any $\tau \geq 0$ and $T \geq 0$, define a space

$$X(\tau - r, T + \tau) = C([\tau - r, T + \tau]; H_\omega^2(\mathbb{R})) \cap L^2([\tau - r, T + \tau]; H_\omega^2(\mathbb{R})) \cap \mathcal{C}_{\text{unif}}[\tau - r, T + \tau]$$

with the norm

$$M_\tau(T)^2 = \sup_{t \in [\tau - r, T + \tau]} (\|u(t)\|_C^2 + \|u(t)\|_{H_\omega^2}^2),$$

and denote $u(t) = u(t, \cdot)$ and $M(T) = M_0(T)$. The estimate of $v(t, \xi)$ in the space $H_\omega^2(\mathbb{R})$ is the following.

Lemma 2 *If $v(t, \xi) \in X(-r, T)$, then*

$$\begin{aligned} & \|v(t)\|_{L_\omega^2}^2 \\ & + \int_0^t \int_{-\infty}^\infty e^{-2\mu(t-s)} [B_{\eta, \mu, \omega}(\xi) - C_1 M(T)] \omega v^2 d\xi ds \\ & \leq C_3 e^{-2\mu t} \left(\|v_0(0)\|_{L_\omega^2}^2 + \int_{-r}^0 \|v_0(s)\|_{L_\omega^2}^2 ds \right), \end{aligned}$$

where

$$\begin{aligned} B_{\eta, \mu, \omega}(\xi) &= A_{\eta, \omega}(\xi) - 2\mu - \frac{f'(0)}{\eta} \partial_2 F(0, 0) \\ & \times (e^{2\mu r} - 1) \int_{-\infty}^\infty J(y) e^{-2\lambda_*(y+cr)} dy, \quad (6) \end{aligned}$$

$$\begin{aligned} A_{\eta, \omega}(\xi) &= -c \frac{\omega'}{\omega} - d \left(\frac{\omega'}{\omega} \right)^2 \\ & - 2\partial_1 F(\xi) - \eta f'(0) \partial_2 F(0, 0) \\ & - \frac{1}{\eta} f'(0) \partial_2 F(0, 0) \int_{-\infty}^\infty J(y) \frac{\omega(\xi + y + cr)}{\omega(\xi)} dy, \end{aligned}$$

and $\mu, \eta, C_i, i = 1, 3$ are positive numbers to be defined later.

Proof: Multiplying (4) by $\omega(\xi)v(t, \xi)e^{2\mu t}$, we have

$$\begin{aligned} & \left(\frac{1}{2} e^{2\mu t} \omega v^2 \right)_t + e^{2\mu t} \left(\frac{1}{2} c \omega v^2 - d \omega v v_\xi \right)_\xi \\ & + e^{2\mu t} d \omega v_\xi^2 + e^{2\mu t} d \omega' v v_\xi \\ & + \left(-\frac{c}{2} \frac{\omega'}{\omega} - \mu \right) e^{2\mu t} \omega v^2 - \partial_1 F(\xi) e^{2\mu t} \omega v^2 \\ & = e^{2\mu t} \omega v \partial_2 F(\xi) \int_{-\infty}^\infty J(y) [f(\phi + v) - f(\phi)] dy \\ & + e^{2\mu t} \omega v Q. \quad (7) \end{aligned}$$

Further,

$$|e^{2\mu t} d \omega' v v_\xi| \leq \frac{1}{2} d e^{2\mu t} \omega v_\xi^2 + \frac{d}{2} \left(\frac{\omega'}{\omega} \right)^2 e^{2\mu t} \omega v^2.$$

Thus (7) is changed into

$$\begin{aligned} & \left(\frac{1}{2} e^{2\mu t} \omega v^2 \right)_t + \frac{1}{2} d e^{2\mu t} \omega v_\xi^2 \\ & + e^{2\mu t} \left(\frac{1}{2} c \omega v^2 - d \omega v v_\xi \right)_\xi \\ & + \left(-\frac{c}{2} \frac{\omega'}{\omega} - \mu - \frac{d}{2} \left(\frac{\omega'}{\omega} \right)^2 - \partial_1 F(\xi) \right) e^{2\mu t} \omega v^2 \\ & \leq e^{2\mu t} \omega v \partial_2 F(\xi) \int_{-\infty}^\infty J(y) [f(\phi + v) - f(\phi)] dy \\ & + e^{2\mu t} \omega v Q. \quad (8) \end{aligned}$$

Integrating (8) over $\mathbb{R} \times [0, t]$ with respect to ξ and t and note that $(\sqrt{\omega}v)|_{\xi=\pm\infty} = (\sqrt{\omega}v_\xi)|_{\xi=\pm\infty} = 0$, we obtain

$$\begin{aligned} & e^{2\mu t} \|v(t)\|_{L_\omega^2(\mathbb{R})}^2 + d \int_0^t e^{2\mu s} \|v_\xi(s)\|_{L_\omega^2(\mathbb{R})}^2 ds \\ & + \int_0^t \int_{-\infty}^\infty \left(-c \frac{\omega'}{\omega} - 2\mu - d \left(\frac{\omega'}{\omega} \right)^2 - 2\partial_1 F(\xi) \right) \\ & \times e^{2\mu s} \omega(\xi) v^2(s, \xi) d\xi ds \\ & \leq \|v_0(0)\|_{L_\omega^2(\mathbb{R})}^2 + 2 \int_0^t \int_{-\infty}^\infty e^{2\mu s} \omega(\xi) v(s, \xi) Q d\xi ds \\ & + 2 \int_0^t \int_{-\infty}^\infty e^{2\mu s} \omega(\xi) v(s, \xi) \partial_2 F(\xi) \\ & \times \int_{-\infty}^\infty J(y) (f(v + \phi) - f(\phi)) dy d\xi ds. \quad (9) \end{aligned}$$

Note that

$$\begin{aligned} & 2|v(s, \xi)| \partial_2 F(\xi) \int_{-\infty}^\infty J(y) |f(v + \phi) - f(\phi)| dy \\ & \leq \partial_2 F(0, 0) f'(0) \int_{-\infty}^\infty J(y) \\ & \times \left(\eta v^2(s, \xi) + \frac{1}{\eta} v^2(s - r, \xi - y - cr) \right) dy, \end{aligned}$$

where $\eta > 0$ will be specified later and

$$\begin{aligned} & \frac{1}{\eta} \int_0^t \int_{-\infty}^\infty e^{2\mu s} \omega(\xi) \\ & \times \int_{-\infty}^\infty J(y) v^2(s - r, \xi - y - cr) dy d\xi ds \\ & \leq \frac{1}{\eta} e^{2\mu r} \int_0^t \int_{-\infty}^\infty \int_{-\infty}^\infty e^{2\mu s} \omega(\xi + y + cr) \\ & \times J(y) v^2(s, \xi) dy d\xi ds \\ & + \frac{1}{\eta} e^{2\mu r} \int_{-r}^0 \int_{-\infty}^\infty \int_{-\infty}^\infty e^{2\mu s} \omega(\xi + y + cr) \\ & \times J(y) v_0^2(s, \xi) dy d\xi ds. \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 & 2 \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \omega(\xi) |v(s, \xi)| \partial_2 F(\xi) \\
 & \quad \times \int_{-\infty}^{\infty} J(y) |f(v + \phi) - f(\phi)| dy d\xi ds \\
 & \leq \eta \partial_2 F(0, 0) f'(0) \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \omega(\xi) v^2(s, \xi) d\xi ds \\
 & + \frac{1}{\eta} \partial_2 F(0, 0) f'(0) e^{2\mu r} \\
 & \times \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(y) e^{2\mu s} \omega(\xi + y + cr) v^2(s, \xi) dy d\xi ds \\
 & \quad + \frac{1}{\eta} \partial_2 F(0, 0) f'(0) e^{2\mu r} \\
 & \times \int_{-r}^0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(y) e^{2\mu s} \omega(\xi + y + cr) v_0^2(s, \xi) dy d\xi ds.
 \end{aligned} \tag{10}$$

For the nonlinearity Q , by applying Taylor's formula to (5), it follows that

$$\begin{aligned}
 Q &= \frac{1}{2} \partial_{11} F(\bar{\phi}, \bar{\phi}) v^2 \\
 & + \partial_{12} F(\bar{\phi}, \bar{\phi}) \left[\int_{-\infty}^{\infty} J(y) [f(v + \phi) - f(\phi)] dy \right] v \\
 & + \frac{1}{2} \partial_{22} F(\bar{\phi}, \bar{\phi}) \left[\int_{-\infty}^{\infty} J(y) [f(v + \phi) - f(\phi)] dy \right]^2,
 \end{aligned} \tag{11}$$

where $\bar{\phi}$ is between ϕ and $\phi + v$ and $\tilde{\phi}$ is between $\int_{-\infty}^{\infty} J(y) f(\phi(\xi - y - cr)) dy$ and $\int_{-\infty}^{\infty} J(y) f(v + \phi) dy$. For the third term on the right-hand side of (11), using Hölder's inequality gives

$$\begin{aligned}
 & \left| \int_{-\infty}^{\infty} J(y) [f(v + \phi) - f(\phi)] dy \right| \\
 & \leq \int_{-\infty}^{\infty} J(y) f'(0) |v(s - r, \xi - y - cr)| dy \\
 & \leq \left(\int_{-\infty}^{\infty} J(y) (f'(0) v(s - r, \xi - y - cr))^2 dy \right)^{1/2}.
 \end{aligned}$$

Because of the definition of $M(T)$ and A_3 , then

$$\begin{aligned}
 & \left| 2 \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \omega(\xi) v(s, \xi) Q d\xi ds \right| \\
 & \leq C_1 M(T) \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \omega(\xi) v^2(s, \xi) d\xi ds \\
 & \quad + C_2 \int_{-r}^0 \int_{-\infty}^{\infty} e^{2\mu s} \omega(\xi) v_0^2(s, \xi) d\xi ds, \tag{12}
 \end{aligned}$$

where

$$\begin{aligned}
 C_1 &= L \left(1 + f'(0) + 2f'(0) e^{2\mu r} \int_{-\infty}^{\infty} J(y) e^{-2\lambda_* y} dy \right), \\
 C_2 &= 2LM(T) f'(0) e^{2\mu r} \int_{-\infty}^{\infty} J(y) e^{-2\lambda_* y} dy
 \end{aligned}$$

and $L = \max\{|\partial_{ij} F(x, y)|\}$ for $(x, y) \in [0, K] \times [0, f(K)]$, $i, j = 1, 2$.

Substituting (10) and (12) into (9), we obtain

$$\begin{aligned}
 & e^{2\mu t} \|v(t)\|_{L^2_\omega}^2 + d \int_0^t e^{2\mu s} \|v_\xi(s)\|_{L^2_\omega}^2 ds \\
 & + \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} (B_{\eta, \mu, \omega}(\xi) - C_1 M(T)) \omega v^2 d\xi ds \\
 & \leq C_3 \left(\|v_0(0)\|_{L^2_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{L^2_\omega}^2 ds \right), \tag{13}
 \end{aligned}$$

where $B_{\eta, \mu, \omega}(\xi)$ is given in (6) and $C_3 = \max\{1, C_2 + (f'(0)/\eta) \partial_2 F(0, 0) e^{2\mu r} \int_{-\infty}^{\infty} J(y) e^{-2\lambda_* y} dy\}$. \square

Lemma 3 Let $\eta = \left(\int_{-\infty}^{\infty} J(y) e^{-2\lambda_*(y+cr)} dy\right)^{1/2}$. Then for $c > \tilde{c}$, we have $A_{\eta, \omega}(\xi) \geq C_4 > 0$ for $\xi \in \mathbb{R}$.

Proof: Since $\omega(\xi) = e^{-2\lambda_*(\xi-x_0)}$, $\omega'(\xi)/\omega(\xi) = -2\lambda_*$, $\omega(\xi + y + cr)/\omega(\xi) = e^{-2\lambda_*(y+cr)}$ and $c > \tilde{c}$,

$$\begin{aligned}
 & A_{\eta, \omega}(\xi) \\
 & \geq 2c\lambda_* - 4d\lambda_*^2 - 2\partial_1 F(0, 0) - \eta \partial_2 F(0, 0) f'(0) \\
 & \quad - \frac{1}{\eta} \partial_2 F(0, 0) f'(0) \int_{-\infty}^{\infty} J(y) e^{-2\lambda_*(y+cr)} dy \\
 & =: C_4 > 0.
 \end{aligned}$$

\square

Lemma 4 Let $v(t, \xi) \in X(-r, T)$ and $\mu_1 > 0$ be the unique root of the equation

$$\begin{aligned}
 & C_4 - 2\mu - \partial_2 F(0, 0) f'(0) (e^{2\mu r} - 1) \\
 & \quad \times \left(\int_{-\infty}^{\infty} J(y) e^{-2\lambda_*(y+cr)} dy \right)^{1/2} = 0.
 \end{aligned}$$

Then for $0 < \mu < \mu_1$, we have

$$\begin{aligned}
 & \|v(t)\|_{L^2_\omega}^2 + \int_0^t e^{-2\mu(t-s)} \|v(s)\|_{L^2_\omega}^2 ds \\
 & \leq C_6 e^{-2\mu t} \left(\|v_0(0)\|_{L^2_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{L^2_\omega}^2 ds \right), \tag{14}
 \end{aligned}$$

where $M(T) \ll 1$ and C_6 will be defined later.

Proof: By Lemma 3 we have that for $0 < \mu < \mu_1$,

$$\begin{aligned}
 B_{\eta,\mu,\omega}(\xi) &\geq C_4 - 2\mu - \partial_2 F(0, 0) f'(0) (e^{2\mu r} - 1) \\
 &\quad \times \left(\int_{-\infty}^{\infty} J(y) e^{-2\lambda_*(y+cr)} dy \right)^{1/2} \\
 &=: C_5 > 0. \tag{15}
 \end{aligned}$$

Substitute (15) into (13) and let $0 < M(T) < C_5/C_1$ be small enough, then

$$\begin{aligned}
 e^{2\mu t} \|v(t)\|_{L^2_\omega}^2 &+ \int_0^t e^{2\mu s} \|v_\xi(s)\|_{L^2_\omega}^2 ds \\
 &+ \int_0^t e^{2\mu s} \|v(s)\|_{L^2_\omega}^2 ds \\
 &\leq C_6 \left(\|v_0(0)\|_{L^2_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{L^2_\omega}^2 ds \right), \tag{16}
 \end{aligned}$$

where $C_6 := C_3 / \min\{1, d, C_5 - C_1 M(T)\}$. □

Lemma 5 Let $v(t, \xi) \in X(-r, T)$. Then for $c > \tilde{c}$,

$$\begin{aligned}
 \|v_\xi(t)\|_{L^2_\omega}^2 &+ \int_0^t e^{-2\mu(t-s)} \|v_\xi(s)\|_{L^2_\omega}^2 ds \\
 &\leq C_{11} e^{-2\mu t} \left(\|v_0(0)\|_{H^1_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{H^1_\omega}^2 ds \right). \tag{17}
 \end{aligned}$$

Proof: By differentiating (4) with ξ and multiplying it by $e^{2\mu t} \omega(\xi) v_\xi(t, \xi)$, we have

$$\begin{aligned}
 &\left(\frac{1}{2} e^{2\mu t} \omega v_\xi^2 \right)_t + \frac{d}{2} e^{2\mu t} \omega v_{\xi\xi}^2 \\
 &\quad + e^{2\mu t} \left(\frac{1}{2} c \omega v_\xi^2 - d \omega v_\xi v_{\xi\xi} \right)_\xi \\
 &\quad + \left(-\frac{d}{2} \left(\frac{\omega'}{\omega} \right)^2 - \frac{c}{2} \frac{\omega'}{\omega} - \mu - \partial_1 F(\xi) \right) e^{2\mu t} \omega v_\xi^2 \\
 &\quad \leq [\partial_1 F(\xi)]_\xi e^{2\mu t} \omega v v_\xi + e^{2\mu t} \omega v_\xi \\
 &\quad \times \left(\partial_2 F(\xi) \int_{-\infty}^{\infty} J(y) [f(\phi + v) - f(\phi)] dy + Q \right)_\xi. \tag{18}
 \end{aligned}$$

By integrating (18) with respect to ξ and t over $\mathbb{R} \times$

$[0, t]$, we obtain

$$\begin{aligned}
 &e^{2\mu t} \|v_\xi(t)\|_{L^2_\omega}^2 + d \int_0^t e^{2\mu s} \|v_{\xi\xi}(s)\|_{L^2_\omega}^2 ds \\
 &\quad + \int_0^t \int_{-\infty}^{\infty} \left\{ -d \left(\frac{\omega'}{\omega} \right)^2 - c \frac{\omega'}{\omega} - 2\mu - 2\partial_1 F(\xi) \right\} \\
 &\quad \quad \times e^{2\mu s} \omega v_\xi^2 d\xi ds \\
 &\leq \int_0^t \int_{-\infty}^{\infty} 2 \{ [\partial_1 F(\xi)]_\xi v + Q_\xi \} e^{2\mu s} \omega v_\xi d\xi ds \\
 &\quad + \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(y) [f(\phi + v) - f(\phi)]_\xi dy \\
 &\quad \quad \times 2\partial_2 F(\xi) e^{2\mu s} \omega v_\xi d\xi ds. \tag{19}
 \end{aligned}$$

Note that

$$\begin{aligned}
 &[\partial_1 F(\xi)]_\xi v \\
 &\quad + \left(\partial_2 F(\xi) \int_{-\infty}^{\infty} J(y) [f(\phi + v) - f(\phi)] dy + Q \right)_\xi \\
 &= \left\{ \partial_1 F(\tilde{\varphi}, \int_{-\infty}^{\infty} J(y) f(\phi + v) dy) v \right\}_\xi - \partial_1 F(\xi) v_\xi^2 \\
 &\quad + \left\{ \partial_2 F(\phi, \tilde{\varphi}) \int_{-\infty}^{\infty} J(y) [f(\phi + v) - f(\phi)] dy \right\}_\xi \\
 &\quad + \partial_2 F(\xi) \left(\int_{-\infty}^{\infty} J(y) [f(\phi + v) - f(\phi)] dy \right)_\xi \\
 &\quad - \partial_2 F(\xi) \left(\int_{-\infty}^{\infty} J(y) [f(\phi + v) - f(\phi)] dy \right)_\xi
 \end{aligned}$$

where $\tilde{\varphi} = \phi + (1 - \theta_1)v$, $\tilde{\varphi} = \theta_2 \int_{-\infty}^{\infty} J(y) f(\phi) dy + (1 - \theta_2) \int_{-\infty}^{\infty} J(y) f(\phi + v) dy$ with $\theta_1, \theta_2 \in [0, 1]$. Hence (19) becomes

$$\begin{aligned}
 &e^{2\mu t} \|v_\xi(t)\|_{L^2_\omega}^2 + d \int_0^t e^{2\mu s} \|v_{\xi\xi}(s)\|_{L^2_\omega}^2 ds \\
 &\quad + \int_0^t \int_{-\infty}^{\infty} B_{\eta,\mu,\omega}(\xi) e^{2\mu s} \omega v_\xi^2 d\xi ds \\
 &\leq C_7 \int_0^t e^{2\mu s} \|v(s)\|_{L^2_\omega}^2 ds + C_8 \int_0^t e^{2\mu s} \|v_\xi(s)\|_{L^2_\omega}^2 ds \\
 &\quad + C_9 \int_{-r}^0 \|v_0(s)\|_{L^2_\omega}^2 ds + C_{10} \int_{-r}^0 \|v_{0\xi}(s)\|_{L^2_\omega}^2 ds. \tag{20}
 \end{aligned}$$

Combining with (16), it holds that

$$\begin{aligned}
 &e^{2\mu t} \|v_\xi(t)\|_{L^2_\omega}^2 + \int_0^t e^{2\mu s} \left(\|v_{\xi\xi}(s)\|_{L^2_\omega}^2 \|v_\xi(s)\|_{L^2_\omega}^2 \right) ds \\
 &\leq C_{11} \left(\|v_0(0)\|_{H^1_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{H^1_\omega}^2 ds \right),
 \end{aligned}$$

where $C_{11} = \frac{\max\{C_6C_7 + C_6C_8 + C_9, C_{10}\}}{\min\{1, d, C_5\}}$. \square

Similarly, we obtain the estimate of $v_{\xi\xi}(t, \xi)$.

Lemma 6 Let $v(t, \xi) \in X(-r, T)$. Then for $c > \tilde{c}$, there exists $C_{12} > 0$ such that

$$\begin{aligned} & \|v_{\xi\xi}(t)\|_{L^2_\omega}^2 + \int_0^t e^{-2\mu(t-s)} \|v_{\xi\xi}(s)\|_{L^2_\omega}^2 ds \\ & \leq C_{12} e^{-2\mu t} \left(\|v_0(0)\|_{H^2_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{H^2_\omega}^2 ds \right), \end{aligned} \quad (21)$$

on condition that $M(T) \ll 1$.

By combining with (14), (17) and (21), we obtain the lemma.

Lemma 7 Let $v(t, \xi) \in X(-r, T)$. Then for $c > \tilde{c}$,

$$\begin{aligned} & \|v(t)\|_{H^2_\omega}^2 + \int_0^t e^{-2\mu(t-s)} \|v(s)\|_{H^2_\omega}^2 ds \\ & \leq C_{13} e^{-2\mu t} \left(\|v_0(0)\|_{H^2_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{H^2_\omega}^2 ds \right), \end{aligned} \quad (22)$$

when $M(T) \ll 1$ and $C_{13} = \max\{C_6, C_{11}, C_{12}\}$.

By applying the Sobolev embedding theorem $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ and noting that $\omega(\xi) \geq 1$ for $\xi \in (-\infty, x_0]$, x_0 will be defined later, we obtain the following decay results.

Lemma 8 Let $v(t, \xi) \in X(-r, T)$. For $t \in [0, T]$,

$$\begin{aligned} & \sup_{\xi \in (-\infty, x_0]} |v(t, \xi)| \leq C_{14} \|v(t)\|_{H^2} \leq C_{14} \|v(t)\|_{H^2_\omega} \\ & \leq C_{15} \left(\|v_0(0)\|_{H^2_\omega}^2 + \int_{-r}^0 \|v_0(s)\|_{H^2_\omega}^2 ds \right)^{1/2} e^{-\mu t}, \end{aligned} \quad (23)$$

where C_{14} is the embedding constant and $C_{15} = C_{14} \sqrt{C_{13}}$.

To prove the exponential decay of $v(t, \xi)$ for $\xi \in [x_0, \infty)$, we first show the result of $v(t, \xi)$ when $\xi = \infty$. For $v(t, \xi) \in X(-r, T)$, $\lim_{\xi \rightarrow \infty} v(t, \xi)$ exists uniformly and $\lim_{\xi \rightarrow \infty} v_\xi(t, \xi) = \lim_{\xi \rightarrow \infty} v_{\xi\xi}(t, \xi) = 0$ uniformly for $t \in [0, T]$. We define $z(t) := v(t, \infty)$, $z_0(s) := v_0(s, \infty)$ for $s \in [-r, 0]$. Furthermore, we verify that $v(t, \xi)$ satisfies

$$\begin{aligned} & v_t(t, \xi) + cv_\xi(t, \xi) - dv_{\xi\xi}(t, \xi) - \partial_1 F(\xi)v(t, \xi) \\ & = \partial_2 F(\xi) \int_{-\infty}^{\infty} J(y)f'(\phi)v dy + \tilde{Q}(v), \\ & v(s, \xi) = v_0(s, \xi), \quad (s, \xi) \in [-r, 0] \times \mathbb{R}, \end{aligned} \quad (24)$$

where

$$F(\xi) = F(\phi(\xi), \int_{-\infty}^{\infty} J(y)f(\phi(\xi - y - cr)) dy),$$

$$\begin{aligned} \tilde{Q}(v) &= F(v + \phi, \int_{-\infty}^{\infty} J(y)f(v + \phi) dy) - F(\xi) \\ &\quad - \partial_1 F(\xi)v(t, \xi) - \partial_2 F(\xi) \int_{-\infty}^{\infty} J(y)f'(\phi)v dy. \end{aligned}$$

By letting $\xi \rightarrow \infty$ to (24), we have

$$\begin{aligned} & z'(t) - \partial_1 F(K, f(K))z(t) \\ & \quad - \partial_2 F(K, f(K))f'(K)z(t-r) = \tilde{Q}(z), \\ & z(s) = z_0(s), \quad s \in [-r, 0]. \end{aligned} \quad (25)$$

Lemma 9 If (A_5) holds and $z(t)$ is the solution of (25), then there exists $C_{16} > 0$ such that

$$|z(t)| \leq C_{16}M(0)e^{-\mu_2 t}, \quad t > 0,$$

when $0 < \mu_2 < -\partial_1 F(K, f(K))$ and $M(0) \ll 1$.

Proof: Let $l_1 = -\partial_1 F(K, f(K)) \geq 0$ and $l_2 = \partial_2 F(K, f(K))f'(K) > 0$. It follows from Lemma 1 that if $l_2 \leq l_1$ with any $r > 0$, or if $l_2 > l_1$ with $0 < r < \bar{r}$, then there exists $C_{16} > 0$ such that

$$|z(t)| \leq C_{16}M(0)e^{-\mu_2 t}, \quad t > 0,$$

with some constant $0 < \mu_2 = \mu_2(l_1, l_2, r) < l_1$, if $M(0) \ll 1$ and \bar{r} is defined in (3). \square

Note that $\lim_{\xi \rightarrow \infty} |e^{\mu_2 t} v(t, \xi) - e^{\mu_2 t} v(t, \infty)| = 0$ uniformly for $t \in [0, T]$. Then for $\epsilon = M(0)$, there exists $x_0 = x_0(M(0)) \gg 1$ such that $|e^{\mu_2 t} v(t, \xi) - e^{\mu_2 t} v(t, \infty)| < \epsilon$ if $\xi \geq x_0$. That is,

$$\begin{aligned} & |e^{\mu_2 t} |v(t, \xi)| - e^{\mu_2 t} |v(t, \infty)|| \\ & \leq |e^{\mu_2 t} v(t, \xi) - e^{\mu_2 t} v(t, \infty)| < \epsilon. \end{aligned}$$

Thus from Lemma 9,

$$e^{\mu_2 t} |v(t, \xi)| \leq C_{16}M(0) + \epsilon, \quad \xi \geq x_0, \quad t \in [0, T].$$

Lemma 10 Let the assumptions defined in Lemma 9 hold. Then there is $x_0 \gg 1$ independent of t such that

$$\sup_{\xi \in [x_0, \infty)} |v(t, \xi)| \leq C_{17}M(0)e^{-\mu_2 t} \quad (26)$$

for $t \in [0, T]$, where $C_{17} = C_{16} + 1$.

Hence we can obtain the following result.

Theorem 3 (a priori estimate) *If assumptions in Theorem 2 hold and $v(t, \xi) \in X(-r, T)$ is a local solution of (4) with a given positive number T , then there are numbers $\delta_1 > 0$, $0 < \mu < \min\{\mu_1, \mu_2\}$ and $C_{18} = \max\{C_{13}, (C_{15} + C_{17})^2\}$ independent of T such that when $M(T) \leq \delta_1$,*

$$\begin{aligned} & \|v(t)\|_C^2 + \|v(t)\|_{H_\omega^2}^2 \\ & \quad + \int_0^t e^{-2\mu(t-s)} \|v(s)\|_{H_\omega^2}^2 ds \\ & \leq C_{18} e^{-2\mu t} \left(\max_{s \in [-r, 0]} \|v_0(s)\|_C^2 + \|v_0(0)\|_{H_\omega^2}^2 \right) \\ & \quad + C_{18} e^{-2\mu t} \int_{-r}^0 \|v_0(s)\|_{H_\omega^2}^2 ds, \quad t \in [0, T]. \end{aligned} \quad (27)$$

Proof: Combining the assumptions with (22), (23), and (26), we obtain (27) directly. \square

STABILITY OF TRAVELLING WAVES

This section is devoted to prove the stability of travelling wave solutions for (1). To obtain the exponential decay estimate, we give the following local existence result.

Proposition 1 (local existence) *For the following problems with the initial value $\tau \geq 0$,*

$$\begin{aligned} & v_t(t, \xi) + cv_\xi(t, \xi) - dv_{\xi\xi}(t, \xi) - \partial_1 F(\xi)v(t, \xi) \\ & = \partial_2 F(\xi) \int_{-\infty}^{\infty} J(y)[f(v + \phi) - f(\phi)] dy + Q, \\ & \quad (t, \xi) \in (\tau, \infty) \times \mathbb{R}, \\ & v(s, \xi) = u_0(s, \xi - cs) - \phi(\xi) =: v_\tau(s, \xi), \\ & \quad (s, \xi) \in [\tau - r, \tau] \times \mathbb{R}. \end{aligned} \quad (28)$$

If $v_\tau(s, \xi) \in X(\tau - r, \tau)$ and $M_\tau(0) \leq \delta_2$ for some constant $\delta_2 > 0$, then there is $t_0 = t_0(\delta_2) > 0$ such that $v(t, \xi) \in X(\tau - r, \tau + t_0)$ and $M_\tau(t_0) \leq \sqrt{2(1+r)}M_\tau(0)$.

Next we give the following result for (4), which suggest Theorem 2 directly.

Theorem 4 (Stability) *Suppose that $(A_1) - (A_5)$ hold. For a travelling wave solution $\phi(\xi)$ of (1) with $c > \tilde{c}$, \tilde{c} is defined in Theorem 2, if $v_0(s, \xi) \in C([-r, 0]; C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \cap L^2([-r, 0]; H_\omega^2(\mathbb{R}))$ and $\lim_{\xi \rightarrow \infty} v_0(s, \xi) =: v_{0,\infty}(s) \in C[-r, 0]$ exists uniformly for $s \in [-r, 0]$, then there are numbers $\delta_0 > 0$ and $\mu > 0$ independent of $v(t, \xi)$ such that when*

$M_\tau(0) \leq \delta_0$, *the unique solution $v(t, x)$ of (4) exists globally and satisfies*

$$\begin{aligned} & \|v(t)\|_C^2 + \|v(t)\|_{H_\omega^2}^2 + \int_0^t e^{-2\mu(t-s)} \|v(s)\|_{H_\omega^2}^2 ds \\ & \leq C_{18} e^{-2\mu t} \left(\max_{s \in [-r, 0]} \|v_0(s)\|_C^2 + \|v_0(0)\|_{H_\omega^2}^2 \right) \\ & \quad + C_{18} e^{-2\mu t} \int_{-r}^0 \|v_0(s)\|_{H_\omega^2}^2 ds, \quad t \in [0, \infty). \end{aligned} \quad (29)$$

Proof: The proof is similar to the previous ones, thus we give the main points as follows. Since positive numbers δ_1 , C_{18} , and μ given in Theorem 3 are independent of T , by choosing $\delta_0 = \delta_1/\sqrt{2(1+r)}$, $\delta_2 = \max\{\sqrt{2C_{18}(1+r)}M(0), \delta_1\}$, and $M(0) \leq \delta_0 < \delta_1 \leq \delta_2$, and by Proposition 1, there is $t_0 = t_0(\delta_2) > 0$ such that $v(t, \xi) \in X(-r, t_0)$ and $M(t_0) \leq \sqrt{2(1+r)}M(0) \leq \delta_1$. Thus for $t \in [0, t_0]$, Theorem 3 gives

$$\begin{aligned} & \|v(t)\|_C^2 + \|v(t)\|_{H_\omega^2}^2 + \int_0^t e^{-2\mu(t-s)} \|v(s)\|_{H_\omega^2}^2 ds \\ & \leq C_{18} e^{-2\mu t} \left(\max_{s \in [-r, 0]} \|v_0(s)\|_C^2 + \|v_0(0)\|_{H_\omega^2}^2 \right) \\ & \quad + C_{18} e^{-2\mu t} \int_{-r}^0 \|v_0(s)\|_{H_\omega^2}^2 ds, \quad t \in [0, t_0]. \end{aligned} \quad (30)$$

When consider (28) at $\tau = t_0$, it clear that $M_{t_0}(0) \leq M(t_0) \leq \delta_1 \leq \delta_2$, and by Theorem 3 we have $v(t, \xi) \in X(-r, 2t_0)$ and $M_{t_0}(t_0) \leq \sqrt{2(1+r)}M_{t_0}(0)$. Combining with

$$\begin{aligned} & M_{t_0}(0) = \sup_{[t_0-r, t_0]} (\|u(t)\|_{C(\mathbb{R})}^2 + \|u(t)\|_{H_\omega^2(\mathbb{R})}^2) \\ & \leq \frac{\delta_1}{\sqrt{2(1+r)}}, \end{aligned}$$

we obtain $M_{t_0}(t_0) \leq \delta_1$. Then

$$\begin{aligned} & M(2t_0) = \sup_{[-r, 2t_0]} (\|u(t)\|_{C(\mathbb{R})}^2 + \|u(t)\|_{H_\omega^2(\mathbb{R})}^2) \\ & \leq \frac{\delta_1}{\sqrt{2(1+r)}} \leq \delta_1. \end{aligned}$$

Consequently, by Theorem 3, we can obtain the exponential decay for $t \in [0, 2t_0]$. By repeating this process, we can prove $v(t, \xi) \in X(-r, \infty)$ and (29) for $t \in [0, \infty)$. \square

APPLICATIONS

In this section we shall apply our consequences obtained in the former sections to some important models arising from practical problems.

Example 1

Consider the case of (1) with the heat kernel $J(y) = \frac{1}{\sqrt{4\pi\rho}} e^{-y^2/4\rho}$, $\rho > 0$. We obtain the following exponential stability of the travelling wave solutions for (1).

Corollary 1 Assume that $(A_1) - (A_5)$ hold. For the given travelling wave solution $\phi(\xi)$ of (1) with the speed $c > \hat{c}$, with \hat{c} satisfying

$$\frac{\hat{c}\lambda_* - d\lambda_*^2 - \partial_1 F(0, 0)}{c_*\lambda_* - d\lambda_*^2 - \partial_1 F(0, 0)} = e^{\lambda_*^2\rho}, \quad (31)$$

if $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \cap L^2([-r, 0]; H_\omega^2(\mathbb{R}))$ and $\lim_{x \rightarrow \infty} [u_0(s, x) - \phi(x + cs)] =: v_{0, \infty} \in C[-r, 0]$ exists uniformly with $s \in [-r, 0]$, then there are numbers $\delta_0 > 0$ and $\mu > 0$ such that when

$$\begin{aligned} \max_{s \in [-r, 0]} \|(u_0 - \phi)(s)\|_C^2 + \|(u_0 - \phi)(0)\|_{H_\omega^2}^2 \\ + \int_{-r}^0 \|(u_0 - \phi)(s)\|_{H_\omega^2}^2 ds \leq \delta_0^2, \end{aligned}$$

the unique solution $u(t, x)$ of (1) and (2) exists globally and it satisfies

$$\begin{aligned} u(t, x) - \phi(x + ct) \in C([-r, \infty); C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \\ \cap L^2([-r, \infty); H_\omega^2(\mathbb{R})) \cap \mathcal{C}_{\text{unif}}[-r, \infty) \end{aligned}$$

and

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu t}, \quad t \geq 0.$$

Remark 2 The condition (31) implies that we may obtain the stability of the slower waves with wave speed near to the critical speed if ρ is sufficiently small.

Example 2

Taking $F(x, y) = -\alpha_1 x + \alpha_2 y$ and $J(y) = \delta(y)$, a Dirac delta function, the equation (1) is reduced to

$$\begin{aligned} u_t(t, x) = du_{xx}(t, x) - \alpha_1 u(t, x) \\ + \alpha_2 f(u(t - r, x)), \quad (32) \end{aligned}$$

where $\alpha_1, \alpha_2 > 0$. We assume the following two assumptions for (32):

- (H_1) $f(K) = \alpha_1 K / \alpha_2$, $f(x) > \alpha_1 x / \alpha_2$, $x \in (0, K)$.
- (H_2) $f'(K) \leq \alpha_1 / \alpha_2$ with any $r > 0$, or $f'(K) > \alpha_1 / \alpha_2$ with $0 < r < \tilde{r}$, where

$$\tilde{r} := \frac{\pi - \arctan \left[\alpha_1^{-1} \sqrt{(\alpha_2 f'(K))^2 - \alpha_1^2} \right]}{\sqrt{(\alpha_2 f'(K))^2 - \alpha_1^2}}.$$

Set $\mathcal{P}(\lambda, c) = d\lambda^2 - c\lambda - \alpha_1 + \alpha_2 f'(0) e^{-\lambda cr}$. Then there exist c_* and λ_* such that $\mathcal{P}(\lambda_*, c_*) = 0$. From Ref. 4, (32) admits a travelling wave solution $\phi(\xi)$, $\xi = x + ct$, for any $c > c_*$ such that

$$\begin{aligned} 0 < \phi(\xi) \leq K, \quad \phi(-\infty) = 0, \\ 0 < \liminf_{\xi \rightarrow \infty} \phi(\xi) \leq \limsup_{\xi \rightarrow \infty} \phi(\xi) \leq K. \end{aligned}$$

Clearly, $\partial_1 F(x, y) = -\alpha_1 < 0$, $\partial_2 F(x, y) = \alpha_2 > 0$, then $\lim_{\xi \rightarrow \infty} \phi(\xi) = K$. We obtain the following stability result of (32).

Corollary 2 Assume that (A_2) , $(H_1) - (H_2)$ hold. For a travelling wave solution $\phi(\xi)$ of (32) with the speed $c > c_*$, if $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \cap L^2([-r, 0]; H_\omega^2(\mathbb{R}))$ and $\lim_{x \rightarrow \infty} [u_0(s, x) - \phi(x + cs)] =: v_{0, \infty} \in C[-r, 0]$ exists uniformly with $s \in [-r, 0]$, then there are $\delta_0 > 0$ and $\mu > 0$ such that when

$$\begin{aligned} \max_{s \in [-r, 0]} \|(u_0 - \phi)(s)\|_C^2 + \|(u_0 - \phi)(0)\|_{H_\omega^2}^2 \\ + \int_{-r}^0 \|(u_0 - \phi)(s)\|_{H_\omega^2}^2 ds \leq \delta_0^2, \end{aligned}$$

the unique solution $u(t, x)$ of (32) and (2) exists globally and satisfies

$$\begin{aligned} u(t, x) - \phi(x + ct) \in C([-r, \infty); C(\mathbb{R}) \cap H_\omega^2(\mathbb{R})) \\ \cap L^2([-r, \infty); H_\omega^2(\mathbb{R})) \cap \mathcal{C}_{\text{unif}}[-r, \infty) \end{aligned}$$

and

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu t}, \quad t \geq 0.$$

Example 3

Consider the following general population model, which derives from the evolution of the mature population with an age structure of a single species¹,

$$\begin{aligned} u_t(t, x) = d\Delta u(t, x) - g(u(t, x)) \\ + h(u(t, x)) \int_{-\infty}^{\infty} J(x - y) f(u(t - r, y)) dy, \quad (33) \end{aligned}$$

where $d > 0, r \geq 0$. We assume the following assumptions.

- (P_1) $h(x), g(x) \in C^2([0, K] \times [0, f(K)], \mathbb{R})$, $g(0) = 0$, $h(K)f(K) = g(K)$, $h(x)f(x) > g(x)$, $h(x) > 0$, $h'(x) \leq 0$, $g'(x) \geq g'(0) \geq 0$ for $x \in (0, K]$, K is defined in A_2 .
- (P_2) There exists $0 < \theta < K$ such that f is increasing on $[0, \theta]$, and $0 < \eta < K$ such that for any $y \in (0, \eta)$, $h(x)y = g(x)$ has a solution $x \in (0, \theta)$.

(P₃) $h(K)f'(K) + h'(K)f(K) - g'(K) \leq 0$ with any time-delay $r > 0$, or $h(K)f'(K) + h'(K)f(K) - g'(K) > 0$ with time-delay $0 < r < \tilde{r}$, when

$$\tilde{r} := \frac{\pi - \arctan((h'(K)f(K) - g'(K))^{-1}\tilde{M})}{\tilde{M}},$$

$$\tilde{M} = \sqrt{(h(K)f'(K))^2 - (h'(K)f(K) - g'(K))^2}.$$

Taking $F(x, y) = -g(x) + h(x)y$ in (1) to obtain (33). It is easy to check that $u \equiv 0$ and $u \equiv K$ are two constant equilibria of (33). Clearly,

$$\begin{aligned} \partial_1 F(x, y) &= -g'(x) + h'(x)y \leq \partial_1 F(0, 0) < 0, \\ 0 < \partial_2 F(x, y) &= h(x) \leq \partial_2 F(0, 0), \\ F(x, y) &\leq -g'(0)x + h(0)y, \end{aligned}$$

for $(x, y) \in [0, K] \times [0, f(K)]$. Set

$$\begin{aligned} \mathcal{P}(\lambda, c) &= d\lambda^2 - c\lambda - g \\ &\quad + h'(0)f'(0) \int_{-\infty}^{\infty} J(y)e^{-\lambda(y+cr)} dy. \end{aligned}$$

Thus there exist c_* and λ_* such that $\mathcal{P}(\lambda_*, c_*) = 0$. From Ref. 1, (33) admits a travelling wave solution $\phi(\xi)$, $\xi = x + ct$, for any $c > c_*$ such that

$$0 < \phi(\xi) \leq K, \quad \phi(-\infty) = 0, \quad \lim_{\xi \rightarrow \infty} \phi(\xi) = K,$$

and obtain the following stability result of (33).

Corollary 3 Assume that $(A_1) - (A_2)$ and $(P_1) - (P_3)$ hold. For a travelling wave solution $\phi(\xi)$ of (33) with the speed $c > \bar{c}$, with \bar{c} satisfying

$$\begin{aligned} \bar{c}\lambda_* - h'(0)f'(0) \left(\int_{-\infty}^{\infty} J(y)e^{-2\lambda_*(y+\bar{c}r)} dy \right)^{1/2} \\ = 2d\lambda_*^2 - g'(0), \end{aligned}$$

if $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; C(\mathbb{R}) \cap H_{\omega}^2(\mathbb{R})) \cap L^2([-r, 0]; H_{\omega}^2(\mathbb{R}))$ and $\lim_{x \rightarrow \infty} [u_0(s, x) - \phi(x + cs)] =: v_{0,\infty} \in C[-r, 0]$ exists uniformly with $s \in [-r, 0]$, then there are $\delta_0 > 0$ and $\mu > 0$ such that when

$$\begin{aligned} \max_{s \in [-r, 0]} \|(u_0 - \phi)(s)\|_C^2 + \|(u_0 - \phi)(0)\|_{H_{\omega}^2}^2 \\ + \int_{-r}^0 \|(u_0 - \phi)(s)\|_{H_{\omega}^2}^2 ds \leq \delta_0^2, \end{aligned}$$

the unique solution $u(t, x)$ of (33) and (2) exists globally and satisfies

$$\begin{aligned} u(t, x) - \phi(x + ct) \in C([-r, \infty); C(\mathbb{R}) \cap H_{\omega}^2(\mathbb{R})) \\ \cap L^2([-r, \infty); H_{\omega}^2(\mathbb{R})) \cap \mathcal{C}_{\text{unif}}[-r, \infty) \end{aligned}$$

and

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu t}, \quad t \geq 0.$$

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