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Endpoint estimates for the commutators of strongly singular integral operators

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ABSTRACT: In this paper, we give sufficient and necessary conditions for the endpoint estimates of the commutators generated by the strongly singular integrals and the BMO function on the extreme case.

KEYWORDS: strongly singular integral, commutator, BMO space, H^1 space, endpoint estimates

MSC2010: 42B20, 42B30

INTRODUCTION

It is well-known that the theory of the strongly Calderón-Zygmund operator was originated from a class of multiplier operators with its symbol given by $e^{i|\xi|^a}/|\xi|^\beta$ with $0 < a < 1$ and $\beta > 0$. In 1972, Fefferman and Stein[1] enlarged the multiplier operators onto a class of convolution operators. Later, Coifman[2] studied a related class of operators in the case $n = 1$.

In 1986, Alvarez and Milman[3] introduced a class of non-convolution operator whose kernel is more singular near the diagonal than those of the classical Calderón-Zygmund operators.

Definition 1 (Ref. 3) Let $T : \mathscr{S} \to \mathscr{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are fulfilled:

(i) T extends to a continuous operator from L^2 into itself.

(ii) T is associated with a certain standard kernel. More precisely, there exists a function $K(x,y)$ continuous on the diagonal of $\mathbb{R}^{2n}$ such that $|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \leqslant C|y-z|^\delta/|x-z|^{n+\delta/\alpha}$ with $2|y-z|^\alpha \leqslant |x-z|$ for some $0<\delta\leqslant 1, 0<\alpha<1$ and $\langle Tf, g\rangle = \int K(x,y)f(y)g(x)\,dx\,dy$ for $f,g\in\mathscr{S}$ with disjoint supports.

(iii) For some $\frac{(1-\alpha)n}{2} \leqslant \beta < \frac{n}{2}$, both operators T and T^* extend to continuous operators from L^q to L^2 with $\frac{1}{q} = \frac{1}{2}+\frac{\beta}{n}$ and $1<q<2$.

(iv) From (iii), we know that T also extends to a continuous operator from L^2 to $L^{q'}$ with $\frac{1}{q'} = 1-\frac{1}{q} = \frac{1}{2}-\frac{\beta}{n}$.

Alvarez and Milman[3] proved that the strongly singular Calderón-Zygmund operator T is bounded from L^∞ to BMO and from L^1 to $L^{1,\infty}$. Here, BMO is the bounded mean oscillation space and its definition can be stated as follows.

Definition 2 (Ref. 4) A function f is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$ if the following sharp maximal function is bounded

$$f^\sharp(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)-f_B|\,dy < \infty,$$

where the supreme is taken over all balls $B\subset\mathbb{R}^n$. Moreover, $f_B = \frac{1}{|B|}\int_B f(x)dx$ and $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_{L^\infty}$.

Then, using the interpolation theory, we know that T is bounded on L^p space with $1<p<\infty$. Moreover, Alvarez and Milman[3] gave the estimates of the sharp maximal function $(Tf)^\sharp$, which implies that the weighted norm estimates for T can be obtained. Thus, by the well-known theorem proved by Alvarez, Bagby, Kurtz and Pérez in Ref. 5, we have the following theorem.

Theorem 1 *(Ref. 6) Suppose that T_b is the commutator generated by the strongly singular Calderón-Zygmund operator T and a BMO function b. Then,*

T_b is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ and the definition of T_b is defined by

$$T_b f(x) = b(x)T(f)(x) - T(bf)(x).$$

Here we would like to mention that the operator T_b was originated from the commutators of the classical C-Z singular integral operators which was proposed by Coifman, Rochberg and Weiss[7] in 1976.

In 1995, Pérez[8] gave a counterexample to show that the commutator of the C-Z singular integral is not bounded from H^1 to L^1. Later, Harboure, Segovia and Torrea[9] gave sufficient and necessary conditions for the endpoint estimates of the commutator generated by the C-Z singular integrals on L^∞ and H^1 spaces.

On the other hand, for the study of commutators generated by the strongly singular integral operators and BMO functions, one may see Refs. 10–12 for more details. However, the sufficient and necessary conditions for the endpoint estimates of T_b is still unknown. In this paper, we will give a positive answer to this question.

Before giving the main results of this paper, we give the atomic decomposition of H^1 space. For more details about H^1 space, one may see Ref. 1.

Definition 3 (Ref. 1) We say a function $a(x)$ is an atom of H^1 if a satisfies the following conditions
(i) $\operatorname{supp}(a) \subset B(x_0, r)$,
(ii) $\|a\|L^\infty \leqslant |B(x_0, r)|^{-1}$,
(iii) $\int a(x)\,dx = 0$.

It is well-known that if a function f belongs to H^1, then it can be written as $f = \sum_{i=1}^{\infty} \lambda_i a_i$ where each a_i is a H^1 atom. Moreover, we have

$$\|f\|_{H^1} \sim \inf\left\{\sum_{i=-\infty}^{+\infty} |a_i|\right\},$$

where the infimum is taken over all decompositions of f.

Our results can be stated as follows.

Theorem 2 *Let α, β and δ be the same as in Definition 1. Suppose that T_b is a commutator generated by the strongly singular integral operator and a BMO function b, then the following two conditions are equivalent:*
(i) *T_b is bounded from L^∞ to BMO.*
(ii) *For any cube $Q = Q(z, r)$, if $r > 1$,*

$$\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|dx\right) \times \left|\int_{(Q(z,2\sqrt{n}r))^c} K(z,y)f(y)dy\right| \leqslant C\|f\|_{L^\infty},$$

where the constant C only depends on n, α, β and δ. On the other hand, for the case $0 < \epsilon \leqslant r \leqslant 1$ with any $\epsilon > 0$,

$$\left(\frac{1}{|Q|}\int_Q |b(x) - b_{Q(z,2\sqrt{n}r^\alpha)}|dx\right) \times \left|\int_{Q(z,2\sqrt{n}r^\alpha)^c} K(z,y)f(y)dy\right| \leqslant C\|f\|_{L^\infty},$$

where the constant C only depends on n, α, β, δ and ϵ.

Theorem 3 *Let α, β and δ be the same as in Definition 1. Suppose that T_b is a commutator generated by the strongly singular integral operator and a BMO function b, then the following two conditions are equivalent:*
(i) *T_b is bounded from H^1 to L^1.*
(ii) *For any H^1 atom $a(x)$ supported on $Q(z, r)$, if $r > 1$,*

$$\left|\int_{Q(z,2\sqrt{n}r)^c} K(x,z)dx \int_Q b(y)a(y)dy\right| \leqslant C,$$

where the constant C only depends on n, α, β and δ. On the other hand, for the case $0 < \epsilon \leqslant r \leqslant 1$ with any $\epsilon > 0$, there is

$$\left|\int_{Q(z,2\sqrt{n}r^\alpha)^c} K(x,z)dx \int_Q b(y)a(y)dy\right| \leqslant C,$$

where the constant C only depends on n, α, β, δ and ϵ.

ENDPOINT ESTIMATES FOR T_b FROM L^∞ TO BMO SPACE.

Before giving the proof of Theorem 2, we give some lemmas that will be very useful throughout this paper.

Lemma 1 *(Ref. 4) Let $1 < p < \infty$, and let $f \in \mathrm{BMO}(\mathbb{R}^n)$, then we have*
(i) $\|f\|_{\mathrm{BMO}} \sim \sup_B \left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\, dx\right)^{1/p}$.
(ii) $\|f\|_{\mathrm{BMO}} \sim \sup_B \inf_{a\in R} \frac{1}{|B|}\int_B |f(x) - a|\, dx$.

Lemma 2 *(Ref. 4) Let $f \in \mathrm{BMO}(\mathbb{R}^n)$, and let $1 \leqslant q < \infty$ and $r_1, r_2 \in \mathbb{R}^+$, then*

$$\left(\frac{1}{|B(z,r_1)|}\int_{B(z,r_1)} |f(x) - f_{B(z,r_2)}|^q dx\right)^{1/q} \leqslant C\left(1 + |\log \frac{r_1}{r_2}|\right)\|f\|_{\mathrm{BMO}}$$

for any $z \in \mathbb{R}^n$.

Now, we are going to give the proof of Theorem 2 and we may consider this problem into the case $r > 1$ and $0 < \epsilon \leqslant r \leqslant 1$.

(i) The case $r > 1$

For any cube $Q = Q(z,r)$ with $r > 1$, splitting f as $f = f_1 + f_2$ with $f_1(x) = f(x)\chi_{Q(z,2\sqrt{n}r)}$. Then using some basic ideas from Ref. 9 (p. 680), we denote

$$\begin{aligned}
\sigma_1(x) &= T_b f_1(x),\\
\sigma_2(x,z) &= (b(x) - b_Q)(Tf_2(x) - Tf_2(z)),\\
\sigma_3(x,z) &= T((b - b_Q)f_2)(z) - T((b - b_Q)f_2)(x),\\
\sigma_4(x,z) &= (b(x) - b_Q)Tf_2(z).
\end{aligned}$$

Thus, we may split $T_b f(x) - (T_b f)_Q$ as

$$\begin{aligned}
T_b f(x) - (T_b f)_Q = \sigma_1(x) &- (\sigma_1)_Q + \sigma_2(x,z)\\
&+ \sigma_4(x,z) - (\sigma_2(\cdot,z))_Q\\
&+ (\sigma_3(x,\cdot))_Q.
\end{aligned}$$

Next, we will give the estimates of $\frac{1}{|Q(z,r)|}\int_{Q(z,r)} |\sigma_i|\, dx$ $(i = 1,2,3)$, respectively. For $\sigma_1(x)$, from (i) of Definition 1 and Theorem 1, we have

$$\begin{aligned}
\frac{1}{|Q(z,r)|}&\int_{Q(z,r)} |\sigma_1(x)| dx\\
&= \frac{1}{|Q(z,r)|}\int_{Q(z,r)} |T_b f_1(x)| dx\\
&\leqslant \frac{1}{|Q(z,r)|}\|T_b f_1\|_2 |Q(z,r)|^{\frac{1}{2}}\\
&\leqslant C\|f_1\|_{L^2} |Q(z,r)|^{-1/2}\\
&\leqslant C\|f\|_{L^\infty} |Q(z,2\sqrt{n}r)|^{1/2} |Q(z,r)|^{-1/2}\\
&\leqslant C\|f\|_{L^\infty}. \qquad (1)
\end{aligned}$$

To estimate σ_2, we give the following estimates. For any $x \in Q(z,r)$ and $y \in Q(z,2\sqrt{n}r)^c$, there is $2|x - z|^\alpha \leqslant 2(\sqrt{n}r)^\alpha \leqslant 2\sqrt{n}r \leqslant |y - z|$ with $r > 1$ and $0 < \alpha < 1$. Then, using (ii) of Definition 1, we have

$$\begin{aligned}
|Tf_2(x) &- Tf_2(z)|\\
&\leqslant \int_{|y-z| \geqslant 2\sqrt{n}r} |K(x,y) - K(z,y)| \times |f(y)| dy\\
&\leqslant C\|f\|_{L^\infty} \int_{|y-z| \geqslant 2\sqrt{n}r} \frac{|x - z|^\delta}{|y - z|^{n + \frac{\delta}{\alpha}}} dy\\
&\leqslant C\|f\|_{L^\infty} r^\delta \int_{2\sqrt{n}r}^{\infty} r^{n-1} r^{-n-\frac{\delta}{\alpha}} dr\\
&\leqslant C\|f\|_{L^\infty} r^{\delta(1-\frac{1}{\alpha})} \leqslant C\|f\|_{L^\infty},
\end{aligned}$$

where the last inequality follows from $r^{\delta(1-\frac{1}{\alpha})} \leqslant r^0 = 1$. Thus, we obtain

$$\frac{1}{|Q(z,r)|}\int_{Q(z,r)} |\sigma_2(x,z)| dx \leqslant C\|b\|_{\mathrm{BMO}} \|f\|_{L^\infty}. \qquad (2)$$

For σ_3, we have

$$\begin{aligned}
|\sigma_3&(x,z)|\\
&= |T((b - b_Q)f_2)(z) - T((b - b_Q)f_2)(x)|\\
&= \left|\int_{\mathbb{R}^n} (b(y) - b_Q)f_2(y)(K(z,y) - K(x,y)) dy\right|\\
&\leqslant \int_{|y-z| \geqslant 2\sqrt{n}r} |b(y) - b_Q||f(y)||K(z,y) - K(x,y)| dy\\
&\leqslant \|f\|_{L^\infty} \int_{|y-z| \geqslant 2\sqrt{n}r} |b(y) - b_Q||K(z,y) - K(x,y)| dy.
\end{aligned}$$

As $0 < \alpha < 1$, we have $2|x - z|^\alpha \leqslant 2(\sqrt{n}r)^\alpha \leqslant 2\sqrt{n}r \leqslant |y - z|$. Then, using (ii) of Definition 1 and

Lemma 2, we get

$$\begin{aligned}|\sigma_3(x,z)| &\leqslant C\|f\|_{L^\infty}\int_{|y-z|\geqslant 2\sqrt{n}r}|b(y)-b_Q| \\ &\quad\times\frac{|x-z|^{\delta}}{|y-z|^{n+\delta/\alpha}}dy \\ &\leqslant C\|f\|_{L^\infty}\sum_{j=0}^{\infty}\int_{2^j\cdot 2\sqrt{n}r\leqslant|y-z|<2^{j+1}\cdot 2\sqrt{n}r} \\ &\qquad |b(y)-b_Q|\frac{|x-z|^{\delta}}{|y-z|^{n+\delta/\alpha}}dy \\ &\leqslant C\|f\|_{L^\infty}\sum_{j=0}^{\infty}\frac{r^{\delta}}{(2^j\cdot 2\sqrt{n}r)^{n+\delta/\alpha}} \\ &\quad\times\int_{2^j\cdot 2\sqrt{n}r\leqslant|y-z|<2^{j+1}\cdot 2\sqrt{n}r}|b(y)-b_Q|dy \\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}\sum_{j=0}^{\infty}\frac{r^{\delta}\cdot(2^{j+1}2\sqrt{n}r)^n}{(2^j\cdot 2\sqrt{n}r)^{n+\delta/\alpha}}(j+1) \\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}r^{\delta(1-\frac{1}{\alpha})}\sum_{j=0}^{\infty}(2^j)^{-\delta/\alpha}(j+1) \\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}r^0 = C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty},\end{aligned}$$

which implies

$$\frac{1}{|Q(z,r)|}\int_{Q(z,r)}|\sigma_3(x,z)|dx \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}. \tag{3}$$

(ii) The case $0<\epsilon\leqslant r\leqslant 1$

For the case $0<\epsilon\leqslant r\leqslant 1$ with any $\epsilon>0$, splitting f as $f=f_1+f_2$ with $f_1=f\chi_{Q(z,2\sqrt{n}r^{\alpha})}$. Then, following some basic ideas from the case $r>1$, we may denote σ_1,σ_2, σ_3 and σ_4 as follows.

$$\begin{aligned}\sigma_1(x) &= T_bf_1(x),\\ \sigma_2(x,z) &= (b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})})(Tf_2(x)-Tf_2(z)),\\ \sigma_3(x,z) &= T((b-b_{Q(z,2\sqrt{n}r^{\alpha})})f_2)(z)\\ &\quad -T((b-b_Q)f_2)(x),\\ \sigma_4(x,z) &= (b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})})Tf_2(z).\end{aligned}$$

Thus, we have

$$\begin{aligned}&T_bf(x)-(T_bf)_{Q(z,2\sqrt{n}r^{\alpha})}\\ &\quad=\sigma_1(x)-(\sigma_1)_{Q(z,2\sqrt{n}r^{\alpha})}+\sigma_2(x,z)+\sigma_4(x,z)\\ &\qquad-(\sigma_2(\cdot,z))_{Q(z,2\sqrt{n}r^{\alpha})}+(\sigma_3(x,\cdot))_{Q(z,2\sqrt{n}r^{\alpha})}.\end{aligned}$$

Next, we will estimate $\frac{1}{|Q|}\int_Q|\sigma_i|dx$ with $i=1,2,3$, respectively. For σ_1, we may give the estimates of $\frac{1}{|Q|}\int_Q|\sigma_1(x)|dx$ as

$$\begin{aligned}&\frac{1}{|Q|}\int_Q|\sigma_1(x)|dx\\ &\quad=\frac{1}{|Q|}\int_Q|T((b(x)-b(\cdot))f_1)(x)|dx\\ &\quad=\frac{1}{|Q|}\int_Q|T(f_1)(x)||b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|dx\\ &\qquad+\frac{1}{|Q|}\int_Q|T((b_{Q(z,2\sqrt{n}r^{\alpha})}-b(\cdot))f_1)(x)|dx\\ &\quad:=I+II.\end{aligned}$$

For I, by using the Hölder inequality, the fact that $0<\epsilon\leqslant r\leqslant 1$, the $L^2\to L^{q'}$ boundedness of T ((iv) of Definition 1) and Lemma 2, we get

$$\begin{aligned}I &= \frac{1}{|Q|}\int_Q|T(f_1)(x)||b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|dx\\ &\leqslant\frac{|Q|^{1/q}}{|Q|}\times\left(\frac{1}{|Q|}\int_Q|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|^qdx\right)^{1/q}\\ &\quad\times\left(\int_Q|Tf_1(x)|^{q'}dx\right)^{1/q'}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f_1\|_{L^2}\frac{|Q|^{1/q}}{|Q|}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}|Q(z,2\sqrt{n}r^{\alpha})|^{1/2}\frac{|Q|^{1/q}}{|Q|}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}r^{n(\frac{\alpha}{2}+\frac{1}{q}-1)}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty},\end{aligned}$$

where the last inequality follows from the fact $0<r\leqslant 1$ and $\frac{\alpha}{2}+\frac{1}{q}-1=\frac{\alpha}{2}-1+\frac{1}{2}+\frac{\beta}{n}\geqslant\frac{\alpha}{2}-\frac{1}{2}+\frac{1}{2}-\frac{\alpha}{2}=0$.

For II, using Lemma 2, the Hölder inequality and the $L^2 \to L^{q'}$ boundedness of T, we have

$$
\begin{aligned}
II &= \frac{1}{|Q|}\int_Q |T((b_{Q(z,2\sqrt{n}r^\alpha)} - b(\cdot))f_1)(x)|dx \\
&\leqslant \frac{1}{|Q|}\|T((b_{Q(z,2\sqrt{n}r^\alpha)} - b(\cdot)f_1(\cdot)))\|_{L^{q'}}|Q|^{1/q} \\
&\leqslant \frac{1}{|Q|}\|(b_{Q(z,2\sqrt{n}r^\alpha)} - b(\cdot))f_1(\cdot)\|_{L^2}|Q|^{1/q} \\
&\leqslant \|f\|_{L^\infty}|Q|^{-1/q'} \\
&\quad\times\left(\int_{Q(z,2\sqrt{n}r^\alpha)} |b_{Q(z,2\sqrt{n}r^\alpha)} - b(x)|^2 dx\right)^{1/2} \\
&\leqslant C\|f\|_{L^\infty}|Q|^{-1/q'}|Q(z,2\sqrt{n}r^\alpha)|^{1/2} \\
&\quad\times\bigg(\frac{1}{|Q(z,2\sqrt{n}r^\alpha)|}\int_{Q(z,2\sqrt{n}r^\alpha)} \\
&\qquad\qquad |b(x) - b_{Q(z,2\sqrt{n}r^\alpha)}|^2 dx\bigg)^{1/2} \\
&\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty} r^{n(\alpha/2+1/q-1)} \\
&= C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}.
\end{aligned}
$$

Combing the above two estimates, we get

$$
\frac{1}{|Q|}\int_Q |\sigma_1(x)|dx \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}. \tag{4}
$$

Next, we give the estimate of $\sigma_2(x,z)$. As $y \in Q(z, 2\sqrt{n}r^\alpha)$ and $x \in Q(z,r)$, there is $2|x-z|^\alpha \leqslant 2\sqrt{n}r^\alpha \leqslant |y-z|$. Thus, we obtain

$$
\begin{aligned}
&|Tf_2(x) - Tf_2(z)| \\
&\leqslant \int_{\mathbb{R}^n} |K(x,y) - K(z,y)|f_2(y)|dy \\
&\leqslant \int_{|y-z|\geqslant 2\sqrt{n}r^\alpha} |K(x,y) - K(z,y)||f_2(y)|dy \\
&\leqslant \|f\|_{L^\infty}\int_{|y-z|\geqslant 2\sqrt{n}r^\alpha} \frac{|x-z|^\delta}{|y-z|^{n+\frac{\delta}{\alpha}}}dy \\
&\leqslant C\|f\|_{L^\infty} r^\delta \int_{2\sqrt{n}r^\alpha}^\infty t^{n-1}t^{-n-\frac{\delta}{\alpha}}dt \leqslant C\|f\|_{L^\infty}.
\end{aligned}
$$

Using Lemma 2 and the fact $0 < \epsilon \leqslant \alpha < 1$, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q |\sigma_2(x,z)|dx \\
&\quad\leqslant C\|f\|_{L^\infty}\frac{1}{|Q|} \times \int_Q |b(x) - b_{Q(z,2\sqrt{n}r^\alpha)}|dx \\
&\quad\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty},
\end{aligned} \tag{5}
$$

where the last inequalities follows from the fact $0 < \epsilon \leqslant r \leqslant 1$.

For $\sigma_3(x,z)$, by the fact $2|x-z|^\alpha \leqslant 2\sqrt{n}r^\alpha \leqslant |y-z|$, Lemma 2 and (ii) of Definition 1, we get

$$
\begin{aligned}
|\sigma_3(x,z)| &= \bigg|\int_{\mathbb{R}^n} (b(y) - b_{Q(z,2\sqrt{n}r^\alpha)}) \\
&\qquad \times f_2(y)(K(z,y) - K(x,y))dy\bigg| \\
&\leqslant \int_{|y-z|\geqslant 2\sqrt{n}r^\alpha} |b(y) - b_{Q(z,2\sqrt{n}r^\alpha)}| \\
&\quad \times |f(y)||K(x,y) - K(z,y)|dy \\
&\leqslant C\|f\|_{L^\infty}\int_{|y-z|\geqslant 2\sqrt{n}r^\alpha} |b(y) - b_{Q(z,2\sqrt{n}r^\alpha)}| \\
&\quad \times \frac{|x-z|^\delta}{|y-z|^{n+\frac{\delta}{\alpha}}}dy \\
&\leqslant C\|f\|_{L^\infty}\sum_{j=0}^\infty \int_{2^j\cdot 2\sqrt{n}r^\alpha \leqslant |y-z| < 2^{j+1}\cdot 2\sqrt{n}r^\alpha} \\
&\qquad |b(y) - b_{Q(z,2\sqrt{n}r^\alpha)}|\frac{|x-z|^\delta}{|y-z|^{n+\frac{\delta}{\alpha}}}dy \\
&\leqslant C\|f\|_{L^\infty}\sum_{j=0}^\infty \frac{r^\delta}{(2^j\cdot 2\sqrt{n}r^\alpha)^{n+\frac{\delta}{\alpha}}} \\
&\qquad \times \int_{2^j\cdot 2\sqrt{n}r^\alpha \leqslant |y-z| < 2^{j+1}\cdot 2\sqrt{n}r^\alpha} \\
&\qquad |b(y) - b_{Q(z,2\sqrt{n}r^\alpha)}|dy \\
&\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty} \\
&\quad \times \sum_{j=0}^\infty \frac{r^\delta(2^{j+1}\cdot 2\sqrt{n}r^\alpha)^n(j+1)}{(2^{j+1}\cdot 2\sqrt{n}r^\alpha)^{n+\frac{\delta}{\alpha}}} \\
&\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}\sum_{j=0}^\infty 2^{-\frac{\delta}{\alpha}(j+1)}(j+1) \\
&\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty},
\end{aligned}
$$

which implies

$$
\frac{1}{|Q|}\int_Q |\sigma_3(x,z)|dx \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}. \tag{6}
$$

Proof of Theorem 2

Now we are ready to give the proof of Theorem 2. For any cube $Q = Q(z,r)$ with $r > 1$, as $\sigma_4(x,z) = T_bf(x) - (T_b(f))_Q - \sigma_1(x) + (\sigma_1)_Q - \sigma_2(x,z) + (\sigma_2(\cdot,z))_Q - (\sigma_3(x,\cdot))_Q$. From (1)-(3), we find that if T_b is bounded from L^∞ to BMO, then it is easy to get

$$
\frac{1}{|Q|}\int_Q |\sigma_4(x,z)|dx \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}. \tag{7}
$$

By the definition of σ_4, (7) is equivalent to

$$\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx \times \left|\int_{(Q(z,2\sqrt{n}r))^c} K(z,y)f(y)dy\right| \leqslant C\|f\|_{L^\infty}. \quad (8)$$

On the other hand, if (8) holds, it is clear that T_b is bounded from L^∞ to BMO.

Similarly, for any cube $Q=Q(z,r)$ with $0<\epsilon\leqslant r\leqslant 1$, using -, it is easy to see that the boundedness of T_b from L^∞ to BMO is equivalent to

$$\left(\frac{1}{|Q|}\int_Q |b(x)-b_{Q(z,2\sqrt{n}r^\alpha)}|dx\right) \times \left|\int_{(Q(z,2\sqrt{n}r^\alpha))^c} K(z,y)f(y)dy\right| \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^\infty}.$$

Thus, the proof of Theorem 2 has been finished.

ENDPOINT ESTIMATES FOR T_b FROM H^1 TO L^1 SPACES.

In this section, we will give the proof of Theorem 3. For any H^1 atom $a(x)$ supported on $Q=Q(z,r)$, we will estimate $T_b a(x)$ in two cases.

(i) The case $r>1$

For the case $r>1$, we may decompose $T_b a(x)$ by

$$\begin{aligned}
T_b a(x) &= T_b a(x)\chi_{Q(z,2\sqrt{n}r)}(x) \\
&\quad + T_b a(x)\chi_{(Q(z,2\sqrt{n}r))^c}(x) \\
&= T_b a(x)\chi_{Q(z,2\sqrt{n}r)}(x) \\
&\quad + T(a(\cdot)(b(\cdot)-b_Q))(x)\chi_{(Q(z,2\sqrt{n}r))^c}(x) \\
&\quad + \chi_{(Q(z,2\sqrt{n}r))^c}(x)T((b(\cdot)-b_Q)a)(x) \\
&= T_b a(x)\chi_{Q(z,2\sqrt{n}r)}(x) \\
&\quad + T(a(\cdot)(b(\cdot)-b_Q))(x)\chi_{(Q(z,2\sqrt{n}r))^c}(x) \\
&\quad + \chi_{(Q(z,2\sqrt{n}r))^c}(x)\int_Q (K(x,y)-K(x,z)) \\
&\quad \times (b(y)-b_Q)a(y)dy + \chi_{(Q(z,2\sqrt{n}r))^c}(x) \\
&\quad \times \int_Q K(x,z)(b(y)-b_Q)a(y)dy \\
&:= \mu_1(x)+\mu_2(x)+\mu_3(x,z)+\mu_4(x,z).
\end{aligned}$$

Then we will give the estimates of $\|\mu_i(\cdot)\|_{L^1}(i=1,2,3)$, respectively. For μ_1, by Theorem 1 and the Hölder inequality, we have

$$\begin{aligned}
\|\mu_1(\cdot)\|_{L^1} &= \|T_b(a)\|_{L^1(Q(z,2\sqrt{n}r))} \\
&= \int_{Q(z,2\sqrt{n}r)} |T_b(a)(x)|dx \\
&\leqslant C|Q(z,2\sqrt{n}r)|^{1/2}\left(\int_{\mathbb{R}^n}|T_b(a)(x)|^2dx\right)^{1/2} \\
&\leqslant Cr^{\frac{n}{2}}\|a\|_{L^2} \\
&\leqslant C|Q|^{1/2}|Q|^{-1/2}\leqslant C. \quad (9)
\end{aligned}$$

For μ_2, as $x\in Q(z,2\sqrt{n}r)^c$ and $y\in Q(z,r)$, there is $2|y-z|^\alpha\leqslant 2\sqrt{n}r^\alpha\leqslant 2\sqrt{n}r\leqslant|x-z|$. Moreover, by the cancellation condition of a and (ii) of Definition 1, we obtain

$$\begin{aligned}
&\|\mu_2(\cdot)\|_{L^1} \\
&= \int_{Q(z,2\sqrt{n}r)^c}(b(x)-b_Q)\int_Q K(x,y)a(y)dydx \\
&= \int_{Q(z,2\sqrt{n}r)^c}(b(x)-b_Q) \\
&\quad \times \int_Q (K(x,y)-K(x,z))a(y)dydx \\
&\leqslant \int_{Q(z,2\sqrt{n}r)^c}|b(x)-b_Q|\left|\int_Q \frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}}|a(y)|dy\right|dx \\
&\leqslant r^\delta\int_Q|a(y)|dy\int_{Q(z,2\sqrt{n}r)^c}\frac{|b(x)-b_Q|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx.
\end{aligned}$$

As

$$\begin{aligned}
&\int_{Q(z,2\sqrt{n}r)^c}\frac{|b(x)-b_Q|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx \\
&\quad \leqslant \sum_{j=1}^\infty\int_{2^jr\leqslant|x-z|<2^{j+1}r}\frac{|b(x)-b_Q|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx \\
&\quad \leqslant \sum_{j=1}^\infty(2^{j+1}r)^{-n-\frac{\delta}{\alpha}}\int_{Q(z,2^{j+1}r)}|b(x)-b_Q|dx \\
&\quad \leqslant \sum_{j=1}^\infty 2^{-(j+1)(n+\frac{\delta}{\alpha})}jr^{-n-\frac{\delta}{\alpha}}(2^{j+1}r)^n\|b\|_{\mathrm{BMO}} \\
&\quad \leqslant C\|b\|_{\mathrm{BMO}}r^{-\frac{\delta}{\alpha}}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|\mu_2(\cdot)\|_{L^1} &\leqslant C\|b\|_{\mathrm{BMO}}r^{\delta(1-\frac{1}{\alpha})}\|a\|_{L^2}|Q|^{1/2} \\
&\leqslant C\|b\|_{\mathrm{BMO}}r^0 = C\|b\|_{\mathrm{BMO}}. \quad (10)
\end{aligned}$$

For μ_3, as $x \in Q(z, 2\sqrt{n}r)^c$ and $y \in Q(z, r)$, there is $2|y-z|^\alpha < 2\sqrt{n}r^\alpha \leqslant 2\sqrt{n}r \leqslant |x-z|$. Thus, using the Hölder inequality and Lemma 1, we obtain

$$\begin{aligned}\|\mu_3(\cdot,z)\|_{L^1} &\leqslant \int_{Q(z,2\sqrt{n}r)^c} \frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}} \\ &\qquad \times |b(y)-b_Q||a(y)|dy\,dx \\ &\leqslant r^\delta \int_Q |a(y)||b(y)-b_Q|dy \\ &\qquad \times \int_{Q(z,2\sqrt{n}r)^c} \frac{1}{|x-z|^{n+\delta/\alpha}}dx \\ &\leqslant Cr^\delta \|a\|_{L^2}\left(\int_Q |b(y)-b_Q|^2 dy\right)^{1/2} \\ &\qquad \times \int_{2\sqrt{n}r}^\infty r^{n-1-n-\frac{\delta}{\alpha}}dr \\ &\leqslant C\|b\|_{\mathrm{BMO}} r^{\delta(1-\frac{1}{\alpha})}|Q|^{-1/2+1/2} \\ &\leqslant C\|b\|_{\mathrm{BMO}}. \end{aligned} \tag{11}$$

The case $r \leqslant 1$

For any H^1 atom $\operatorname{supp}(a) \subset Q = Q(z, r)$ with $0 < \epsilon \leqslant r \leqslant 1$, we may decompose $T_b a(x)$ as

$$\begin{aligned} &T_b a(x) \\ &\quad = T_b a(x)\chi_{Q(z,2\sqrt{n}r^\alpha)}(x) + T_b a(x)\chi_{(Q(z,2\sqrt{n}r^\alpha))^c}(x) \\ &\quad = T_b a(x)\chi_{Q(z,2\sqrt{n}r^\alpha)}(x) \\ &\qquad + T(a(\cdot)(b(\cdot)-b_{Q(z,2\sqrt{n}r^\alpha)}))(x)\chi_{(Q(z,2\sqrt{n}r^\alpha))^c}(x) \\ &\qquad + \chi_{(Q(z,2\sqrt{n}r^\alpha))^c}(x)T((b_{Q(z,2\sqrt{n}r^\alpha)}-b(\cdot))a)(x) \\ &\quad = T_b a(x)\chi_{Q(z,2\sqrt{n}r^\alpha)}(x) \\ &\qquad + T(a(\cdot)(b(\cdot)-b_{Q(z,2\sqrt{n}r^\alpha)}))(x)\chi_{(Q(z,2\sqrt{n}r^\alpha))^c}(x) \\ &\qquad + \chi_{(Q(z,2\sqrt{n}r^\alpha))^c}(x)\int_Q (K(x,y)-K(x,z)) \\ &\qquad\qquad \times (b(y)-b_{Q(z,2\sqrt{n}r^\alpha)})a(y)dy \\ &\qquad + \chi_{(Q(z,2\sqrt{n}r^\alpha))^c}(x) \\ &\qquad \times \int_Q K(x,z)(b(y)-b_{Q(z,2\sqrt{n}r^\alpha)})a(y)dy \\ &\quad := \mu_1(x)+\mu_2(x)+\mu_3(x,z)+\mu_4(x,z). \end{aligned}$$

Next, we will give the estimates of $\|\mu_i\|_{L^1}$ $(i = 1, 2, 3)$ respectively.

$$\begin{aligned} &\|\mu_1(\cdot)\|_{L^1} \\ &\quad = \int_{Q(z,2\sqrt{n}r^\alpha)} |T((b(x)-b(\cdot))a(\cdot))(x)dx|\,dx \\ &\quad \leqslant \int_{Q(z,2\sqrt{n}r^\alpha)} |Ta(x)||b(x)-b_{Q(z,2\sqrt{n}r^\alpha)}|dx \\ &\qquad + \int_{Q(z,2\sqrt{n}r^\alpha)} |T((b(\cdot)-b_{Q(z,2\sqrt{n}r^\alpha)})a(\cdot))(x)|dx \\ &\quad := I + II. \end{aligned}$$

For I, by the $L^q \to L^2$ boundedness of T ((iii) of Definition 1), we have

$$\begin{aligned} I &= \int_{Q(z,2\sqrt{n}r^\alpha)} |Ta(x)||b(x)-b_{Q(z,2\sqrt{n}r^\alpha)}|dx \\ &\leqslant \left(\int_{Q(z,2\sqrt{n}r^\alpha)} |b(x)-b_{Q(z,2\sqrt{n}r^\alpha)}|^2 dx\right)^{1/2} \\ &\quad \times \left(\int_{Q(z,2\sqrt{n}r^\alpha)} |Ta(x)|^2 dx\right)^{1/2} \\ &\leqslant C\|b\|_{\mathrm{BMO}}|Q(z,2\sqrt{n}r^\alpha)|^{1/2}\|a\|_{L^q} \\ &\leqslant C\|b\|_{\mathrm{BMO}} r^{n(\frac{\alpha}{2}+1/q-1)}. \end{aligned}$$

As $\frac{\alpha}{2}+\frac{1}{q}-1 = \frac{\alpha}{2}-1+\frac{1}{2}+\frac{\beta}{n} \geqslant \frac{\alpha}{2}-\frac{1}{2}+\frac{1}{2}-\frac{\alpha}{2} = 0$ and $0 < r \leqslant 1$, we may get

$$I \leqslant C\|b\|_{\mathrm{BMO}} r^0 = C\|b\|_{\mathrm{BMO}}.$$

For II, using the Hölder inequality, Lemma 2 and the $L^q \to L^2 (1 < q < 2)$ boundedness of T, we have

$$\begin{aligned} II &= \int_{Q(z,2\sqrt{n}r^\alpha)} |T((b(\cdot)-b_{Q(z,2\sqrt{n}r^\alpha)})a(\cdot))(x)|dx \\ &\leqslant \left(\int_{Q(z,2\sqrt{n}r^\alpha)} \left|T((b(\cdot)-b_{Q(z,2\sqrt{n}r^\alpha)})a(\cdot))(x)\right|^2 dx\right)^{1/2} \\ &\quad \times |Q(z,2\sqrt{n}r^\alpha)|^{1/2} \\ &\leqslant C\|(b(\cdot)-b_{Q(z,2\sqrt{n}r^\alpha)})a(\cdot)\|_{L^q} r^{\frac{\alpha}{2}n} \\ &\leqslant Cr^{\frac{\alpha}{2}n}\|a\|_{L^2}\left(\int_{Q(z,r)} |b(x)-b_{Q(z,2\sqrt{n}r^\alpha)}|^{\frac{2q}{2-q}} dx\right)^{\frac{2-q}{2q}}. \end{aligned}$$

Recall $\frac{\alpha}{2}+\frac{1}{q}-1\geqslant 0$. Then, using the fact $0<\epsilon\leqslant r\leqslant 1$ and Lemma 2, we get

$$\begin{aligned}II &\leqslant r^{\frac{\alpha}{2}n}\|a\|_{L^2}\left(\frac{1}{|Q|}\int_Q |b(y)-b_{Q(z,2\sqrt{n}r^{\alpha})}|^{\frac{2q}{2-q}}dy\right)^{\frac{2-q}{2q}}\\&\quad\times|Q|^{(1-\frac{q}{2})\frac{1}{q}}\\&\leqslant C\|b\|_{\mathrm{BMO}}r^{\frac{\alpha}{2}n}r^{n(1-\frac{q}{2})\frac{1}{q}}r^{-\frac{n}{2}}\\&=C\|b\|_{\mathrm{BMO}}r^{n(\frac{\alpha}{2}+1/q-1)}\leqslant C\|b\|_{\mathrm{BMO}}.\end{aligned}$$

Combining the above two estimates, we obtain

$$\|\mu_1(\cdot)\|_{L^1}\leqslant C\|b\|_{\mathrm{BMO}}. \tag{12}$$

For μ_2, by the fact $y\in Q(z,r)$ and $x\in Q(z,2\sqrt{n}r^{\alpha})$, there is $2|y-z|^{\alpha}\leqslant 2\sqrt{n}r^{\alpha}\leqslant|x-z|$. Thus, we get

$$\begin{aligned}\|\mu_2(\cdot)\|_{L^1}&\leqslant\int_{Q(z,2\sqrt{n}r^{\alpha})^c}|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|\\&\quad\times\int_{Q(z,r)}\frac{|y-z|^{\delta}|a(y)|}{|x-z|^{n+\frac{\delta}{\alpha}}}dydx\\&\leqslant r^{\delta}\int_{Q(z,r)}|a(y)|dy\\&\quad\times\int_{Q(z,2\sqrt{n}r^{\alpha})^c}\frac{|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx.\end{aligned}$$

As

$$\begin{aligned}&\int_{Q(z,2\sqrt{n}r^{\alpha})^c}\frac{|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx\\&=\int_{|x-z|\geqslant 2\sqrt{n}r^{\alpha}}\frac{|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx\\&\leqslant\sum_{j=1}^{\infty}\int_{2^j\sqrt{n}r^{\alpha}\leqslant|x-z|<2^{j+1}\sqrt{n}r^{\alpha}}\\&\quad\frac{|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|}{|x-z|^{n+\frac{\delta}{\alpha}}}dx\\&\leqslant\sum_{j=1}^{\infty}(2^{j+1}\sqrt{n}r^{\alpha})^{-n-\frac{\delta}{\alpha}}\int_{Q(z,2^{j+1}\sqrt{n}r^{\alpha})}\\&\quad|b(x)-b_{Q(z,2\sqrt{n}r^{\alpha})}|dx\\&\leqslant C\|b\|_{\mathrm{BMO}}\sum_{j=1}^{\infty}(2^{j+1}\sqrt{n}r^{\alpha})^{-n-\frac{\delta}{\alpha}}(j+1)\\&\quad\times(2^{j+1}\sqrt{n}r^{\alpha})^{n}\\&=C\|b\|_{\mathrm{BMO}}\sum_{j=1}^{\infty}(2^{j+1}\sqrt{n}r^{\alpha})^{-\frac{\delta}{\alpha}}(j+1),\end{aligned}$$

we may get

$$\begin{aligned}\|\mu_2(\cdot)\|_{L^1}&\leqslant C\|b\|_{\mathrm{BMO}}r^{\delta}\|a\|_{L^2}|Q|^{1/2}\\&\quad\times\sum_{j=1}^{\infty}(2^{j+1}\sqrt{n}r^{\alpha})^{-\frac{\delta}{\alpha}}(j+1)\leqslant C\|b\|_{\mathrm{BMO}}.\end{aligned} \tag{13}$$

For $\mu_3(x,z)$, note that $2|y-z|^{\alpha}\leqslant 2\sqrt{n}r^{\alpha}\leqslant|z-x|$, the fact $0<\epsilon\leqslant r\leqslant 1$ with any $\epsilon>0$ and $0<\alpha<1$. Then using (ii) of Definition 1 and Lemma 2, we get

$$\begin{aligned}&\|\mu_3(\cdot,z)\|_{L^1}\\&=\int_{Q(z,2\sqrt{n}r^{\alpha})^c}\int_Q|K(x,y)-K(x,z)|\\&\quad\times|b(y)-b_{Q(z,2\sqrt{n}r^{\alpha})}||a(y)|dydx\\&\leqslant\int_{Q(z,2\sqrt{n}r^{\alpha})^c}\int_Q\frac{|y-z|^{\delta}}{|z-x|^{n+\frac{\delta}{\alpha}}}\\&\quad\times|b(y)-b_{Q(z,2\sqrt{n}r^{\alpha})}||a(y)|dydx\\&\leqslant r^{\delta}\int_Q|a(y)||b(y)-b_{Q(z,2\sqrt{n}r^{\alpha})}|dy\\&\quad\times\int_{Q(z,2\sqrt{n}r^{\alpha})^c}\frac{1}{|z-x|^{n+\frac{\delta}{\alpha}}}dx\\&\leqslant Cr^{\delta}\|a\|_{L^2}|Q|^{1/2}\\&\quad\times\left(\frac{1}{|Q|}\int_Q|b(y)-b_{Q(z,2\sqrt{n}r^{\alpha})}|^2dy\right)^{1/2}\\&\quad\times\int_{2\sqrt{n}r^{\alpha}}^{\infty}t^{n-1-n-\frac{\delta}{\alpha}}dt\\&\leqslant Cr^{\delta}\|a\|_{L^2}|Q|^{1/2}\|b\|_{\mathrm{BMO}}\times\int_{2\sqrt{n}r^{\alpha}}^{\infty}t^{n-1-n-\frac{\delta}{\alpha}}dt\\&\leqslant C\|b\|_{\mathrm{BMO}}.\end{aligned} \tag{14}$$

Proof of Theorem 3

Finally, we will give the proof of Theorem 3. From Definition 3, we know that for any $f\in H^1$, f can be decomposed by

$$f=\sum_j\lambda_j a_j,$$

where a_j is an H^1 atom. By using the main results of Ref. 13, we know that the boundedness of T_b from H^1 to L^1 is equivalent to

$$\|T_b a\|_{L^1}\leqslant C, \tag{15}$$

where a is an H^1 atom supported on $Q=Q(z,r)$.

For case $r > 1$. By the definition of $\mu_4(x,z)$, the estimates of μ_i $(i = 1,2,3)$ and (15), we know that if T_b is bounded from H^1 to L^1. Thus, we obtain

$$\left| \int_{Q(z,2\sqrt{n}r)^c} K(x,z)dx \int_Q (b_Q - b(y))a(y)dy \right| \leqslant C. \tag{16}$$

By the cancellation conditions of a, (16) is equivalent to

$$\left| \int_{Q(z,2\sqrt{n}r)^c} K(x,z)dx \int_Q b(y)a(y)dy \right| \leqslant C. \tag{17}$$

On the other hand, if (17) holds, combining (9)-(12), we may get (15), which is equivalent to the fact $T_b f(x)$ is bounded from H^1 to L^1.

Similarly for the case $0 < \epsilon \leqslant r \leqslant 1$, we know that the boundedness of T_b from H^1 to L^1 is also equivalent to

$$\left| \int_{Q(z,2\sqrt{n}r^\alpha)^c} K(x,z)dx \int_Q b(y)a(y)dy \right| \leqslant C.$$

Consequently, the proof of Theorem 3 has been finished.

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