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On a generalization of transformation semigroups that preserve equivalences

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ABSTRACT: Let $T(X)$ be the full transformation semigroup on a nonempty set X. For an equivalence relation σ on X, Pei introduced and studied the subsemigroup of $T(X)$ defined by $T(X,\sigma)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma$ implies $(x\alpha,y\alpha)\in\sigma\}$, which is called a transformation semigroup preserving the equivalence σ. In this paper, for two equivalence relations σ, ρ with $\rho\subseteq\sigma$ on a nonempty set X, we introduce the subsemigroup $T(X,\sigma,\rho)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma$ implies $(x\alpha,y\alpha)\in\rho\}$ of $T(X)$ which generalizes the notation of the subsemigroup $T(X,\sigma)$ of $T(X)$. A necessary and sufficient condition under which $T(X,\sigma,\rho)$ is a BQ-semigroup (a semigroup whose bi-ideals and quasi-ideals coincide) is given. We also prove that $T(X,\sigma)$ of $T(X)$ can be embedded into a semigroup of $T(Y,Z)=\{\alpha\in T(Y):Y\alpha\subseteq Z\}$ for some sets Y and Z with $Z\subseteq Y$.

KEYWORDS: bi-ideals, quasi-ideals, BQ-semigroup

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INTRODUCTION

For a nonempty set X, let $T(X)$ be the full transformations semigroup on X, i.e., $T(X)$ is the semigroup under composition of all mappings $\alpha:X\to X$. Miller and Doss[1] proved that $T(X)$ is a regular semigroup and described its Green's relations. It is well known that every semigroup is isomorphic to a subsemigroup of some full transformation semigroups. Hence in order to study structure of semigroups, it suffices to consider in subsemigroups of $T(X)$.

Let σ be an equivalence relation on a nonempty set X. Pei[2] has studied a family of subsemigroups of $T(X)$ determined by σ, namely,

$$T(X,\sigma)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma \text{ implies } (x\alpha,y\alpha)\in\sigma\}.$$

It is clear that if $\sigma\in\{I_X,X\times X\}$, where I_X is the identity relation on X, then $T(X,\sigma)=T(X)$. He discussed regularity of elements and Green's relations for $T(X,\sigma)$. Mendes-Gonçalves and Sullivan[3] introduced a subsemigroup of $T(X)$ defined by

$$E(X,\sigma)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma \text{ implies } x\alpha=y\alpha\}$$

and call it the *semigroup of transformations restricted by an equivalence* σ. Observe that $E(X,\sigma)$ is a subsemigroup of $T(X,\sigma)$. They also characterized Green's relations on the largest regular subsemigroup of $E(X,\sigma)$ and showed that if $|X|\geqslant 2$ and $\sigma\neq I_X$, then $E(X,\sigma)$ is not isomorphic to $T(Z)$ for any set Z.

Let σ and ρ be equivalence relations on a set X with $\rho\subseteq\sigma$. We define a generalization of $T(X,\sigma)$ as follows:

$$T(X,\sigma,\rho)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma \text{ implies } (x\alpha,y\alpha)\in\rho\}.$$

It is easy to see that $T(X,\sigma,\rho)$ is a subsemigroup of $T(X)$. Notice that the identity mapping need not be in $T(X,\sigma,\rho)$. If $\sigma=I_X$ or $\rho=X\times X$, then $T(X,\sigma,\rho)$ contains the identity mapping on X. And if $\rho=I_X$, then $T(X,\sigma,\rho)$ is a right ideal of $T(X)$.

The relationships between the semigroups $T(X,\sigma,\rho)$ and $E(X,\sigma)$, $T(X,\sigma)$ and $T(X)$ are now described.

Proposition 1 *The following statements hold.*
(i) $E(X,\sigma)\subseteq T(X,\sigma,\rho)\subseteq T(X,\sigma)$.
(ii) $T(X,\sigma,\rho)=E(X,\sigma)$ *if and only if* $\rho=I_X$.
(iii) $T(X,\sigma,\rho)=T(X,\sigma)$ *if and only if* $\sigma=\rho$.
(iv) $T(X,\sigma,\rho)=T(X)$ *if and only if* $\sigma=I_X$ *or* $\rho=X\times X$.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ\cap QS\subseteq Q$, and by a *bi-*

ideal of S we mean a subsemigroup B of S such that $BSB \subseteq B$. Quasi-ideals are a generalization of left ideals and right ideals and bi-ideals are a generalization of quasi-ideals. A *BQ-semigroup* is a semigroup S whose bi-ideals and quasi-ideals coincide. It is known that regular semigroups[4], left (or right) simple semigroups[5], and left (or right) 0-simple semigroups[5] are *BQ*-semigroups.

For a nonempty subset A of a semigroup S, $(A)_{\mathrm{q}}$ and $(A)_{\mathrm{b}}$ denote, respectively, the quasi-ideal and the bi-ideal of S generated by A, that is, $(A)_{\mathrm{q}}$ is the intersection of all quasi-ideals of S containing A and $(A)_{\mathrm{b}}$ is the intersection of all bi-ideals of S containing A [6].

Proposition 2 (Ref. 7) *For a nonempty subset A of a semigroup S,*

$$(A)_{\mathrm{q}} = A \cup (SA \cap AS),\ (A)_{\mathrm{b}} = A \cup A^2 \cup ASA.$$

Calais[8] gave a characterization of the *BQ*-semigroups as follows.

Proposition 3 (Ref. 8) *A semigroup S is a BQ-semigroup if and only if* $(x,y)_{\mathrm{b}} = (x,y)_{\mathrm{q}}$ *for all* x, $y \in S$.

Let Y be a fixed nonempty subset of X. Symons[9] considered the subsemigroup of $T(X)$ defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

and described all the automorphisms of this semigroup. Furthermore, he determined when the two semigroups of this type are isomorphic. Nenthein et al[10] characterized regular elements of $T(X,Y)$ and determined the numbers of regular elements in $T(X,Y)$ for a finite set X. It was also proved that $T(X,Y)$ is a *BQ*-semigroup[11]. Sanwong and Sommanee[12] described $T(X,Y)$ to be regular and determined Green's relations on $T(X,Y)$. They also obtained a class of maximal inverse subsemigroups of $T(X,Y)$.

In this paper, we first prove that $T(X,\sigma,\rho)$ is a *BQ*-semigroup in terms of equivalence relations. Secondly, we show that the semigroup $T(X,\sigma,\rho)$ can be embeddable in $T(Y,Z)$ for some sets Y,Z with $Z \subseteq Y$ and prove that if $\sigma = I_X$ or $\rho = X \times X$, then $T(X,\sigma,\rho) \cong T(Y,Z)$ for some sets Y,Z with $Z \subseteq Y$.

In the remainder of this paper, let σ and ρ be equivalence relations on a set X such that $\rho \subseteq \sigma$.

MAIN RESULTS

Firstly, we characterize when $T(X,\sigma,\rho)$ is a *BQ*-semigroup in terms of equivalences. The following lemmas are needed.

Lemma 1 *Let* $\alpha \in T(X)$. *Then* $\alpha \in T(X,\sigma,\rho)$ *if and only if for each* $A \in X/\sigma$ *there exists* $B \in X/\rho$ *such that* $A\alpha \subseteq B$.

Proof: Suppose that $\alpha \in T(X,\sigma,\rho)$. Let $A \in X/\sigma$ and $a \in A$. Then there exists $B \in X/\rho$ such that $a\alpha \in B$. Let $y \in A\alpha$. Then $x\alpha = y$ for some $x \in A$. Since $(a,x) \in \sigma$ and $\alpha \in T(X,\sigma,\rho)$, we have $(a\alpha, y) = (a\alpha, x\alpha) \in \rho$. This means that $y \in B$. Hence $A\alpha \subseteq B$.

Conversely, suppose that for each $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A\alpha \subseteq B$. Let x, $y \in X$ be such that $(x,y) \in \sigma$. Then $x, y \in A$ for some $A \in X/\sigma$. By assumption, there exists $B \in X/\rho$ such that $x\alpha$, $y\alpha \in A\alpha \subseteq B$. It follows that $(x\alpha, y\alpha) \in \rho$. Hence $\alpha \in T(X,\sigma,\rho)$, as required. □

Lemma 2 (Ref. 11) *Every bi-ideal of a regular semigroup is a BQ-semigroup.*

As was mentioned, if $\rho = I_X$, then $T(X,\sigma,\rho)$ is a right ideal of $T(X)$. Hence $T(X,\sigma,\rho)$ is a bi-ideal of $T(X)$ if $\rho = I_X$. From Lemma 2 we have the following result.

Corollary 1 *If* $\rho = I_X$, *then* $T(X,\sigma,\rho)$ *is a BQ-semigroup.*

Proposition 4 *Let* $\alpha \in T(X,\sigma,\rho)$. *If for each* $A \in X/\sigma$ *there exists* $B \in X/\sigma$ *such that* $A \cap X\alpha \subseteq B\alpha$, *then* $(\alpha)_{\mathrm{b}} = (\alpha)_{\mathrm{q}}$.

Proof: Suppose that for each $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$. Let $\beta \in (\alpha)_{\mathrm{q}}$. If $\beta = \alpha$, then $\beta \in (\alpha)_{\mathrm{b}}$. Assume that $\beta \neq \alpha$. Then $\beta = \alpha\gamma = \lambda\alpha$ for some γ, $\lambda \in T(X,\sigma,\rho)$. Let $A \in X/\sigma$ be such that $A \cap X\alpha \neq \varnothing$. Then $A \cap X\alpha \subseteq B_A$ for some fixed $B_A \in X/\sigma$. For each $y \in A \cap X\alpha$, we choose and fix $a_y \in B_A$ such that $a_y\alpha = y$. For fixed $b_A \in B_A$ and define $\mu_A : A \to X$ by

$$x\mu_A = \begin{cases} a_x\lambda, & x \in X\alpha, \\ b_A\lambda, & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by

$$\mu|_A = \begin{cases} \mu_A, & A \cap X\alpha \neq \varnothing, \\ c_A, & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ and c_A is a constant map from A into X. Since X/σ is a partition of X, μ is well-defined.

For each $A \in X/\sigma$ with $A \cap X\alpha \neq \varnothing$, by Lemma 1 we have that $A\mu_A \subseteq B_A\lambda \subseteq C$ for some $C \in X/\rho$. It follows from Lemma 1 that $\mu \in T(X,\sigma,\rho)$. Let $x \in X$. Then $x\alpha \in A$ for some $A \in X/\sigma$. Since $\beta = \alpha\gamma = \lambda\alpha$, we deduce that

$$x\alpha\mu\alpha = x\alpha\mu_A\alpha = a_{x\alpha}\lambda\alpha = a_{x\alpha}\alpha\gamma = x\alpha\gamma = x\beta.$$

This means that $\beta = \alpha\mu\alpha$ and so $\beta \in (\alpha)_{\mathrm{b}}$. Hence $(\alpha)_{\mathrm{q}} \subseteq (\alpha)_{\mathrm{b}}$. We conclude that $(\alpha)_{\mathrm{q}} = (\alpha)_{\mathrm{b}}$. □

As a consequence of Proposition 4, the following result follows readily.

Corollary 2 *If $\sigma = X \times X$, then $(\alpha)_{\mathrm{b}} = (\alpha)_{\mathrm{q}}$ for all $\alpha \in T(X,\sigma,\rho)$.*

The following theorem characterizes when $T(X,\sigma,\rho)$ is a BQ-semigroup.

Theorem 1 *$T(X,\sigma,\rho)$ is a BQ-semigroup if and only if $\sigma = X \times X$ or $\sigma = I_X$ or $\rho = X \times X$ or $\rho = I_X$.*

Proof: Suppose that $\sigma, \rho \notin \{X \times X, I_X\}$. Since $\rho \neq I_X$, there exist distinct elements $a, b \in X$ such that $(a,b) \in \rho$. It follows from $\rho \subseteq \sigma$ that $a, b \in A$ for some $A \in X/\sigma$. Since $\sigma \neq X \times X$, there is $B \in X/\sigma$ such that $A \neq B$. Let $c \in B$. Define $\alpha, \beta, \gamma : X \to X$ by

$$x\alpha = \begin{cases} a, & x \in A, \\ b, & \text{otherwise}, \end{cases}$$

$$x\beta = \begin{cases} a, & x = b, \\ b, & \text{otherwise}, \end{cases}$$

$$x\gamma = \begin{cases} c, & x \in A, \\ b, & \text{otherwise}. \end{cases}$$

Clearly, α, $\gamma \in T(X,\sigma,\rho)$. Since $(a,b) \in \rho$, $\beta \in T(X,\sigma,\rho)$. We will show that $\alpha\beta = \gamma\alpha$. Let $x \in X$. If $x \in A$, then $x\alpha\beta = a\beta = b = c\alpha = x\gamma\alpha$. If $x \notin A$, then $x\alpha\beta = b\beta = a = b\alpha = x\gamma\alpha$. This means that $\alpha\beta = \gamma\alpha \in (\alpha)_{\mathrm{q}}$. Suppose that $(\alpha)_{\mathrm{q}} = (\alpha)_{\mathrm{b}}$. Since $a\alpha\beta = a\beta = b \neq a = a\alpha$ and $a\alpha\beta = b \neq a = a\alpha = a\alpha\alpha$, it follows that $\alpha\beta \neq \alpha$ and $\alpha\beta \neq \alpha^2$. Since $(\alpha)_{\mathrm{q}} = (\alpha)_{\mathrm{b}} = \{\alpha, \alpha^2\} \cup \alpha T(X,\sigma,\rho)\alpha$, there exists $\mu \in T(X,\sigma,\rho)$ such that $\alpha\beta = \alpha\mu\alpha$. Hence $b = a\alpha\beta = a\alpha\mu\alpha = a\mu\alpha$ and $a = c\alpha\beta = c\alpha\mu\alpha = b\mu\alpha$. It follows that $a\mu \in (a\mu\alpha)\alpha^{-1} = b\alpha^{-1}$ and $b\mu \in (b\mu\alpha)\alpha^{-1} = a\alpha^{-1}$. Since $\mu \in T(X,\sigma,\rho)$ and $(a,b) \in \rho \subseteq \sigma$, we deduce that $(a\mu, b\mu) \in \rho$. Then there is $C \in X/\rho$ such that $a\mu$, $b\mu \in C$. Thus $C \cap a\alpha^{-1} \neq \varnothing$ and $C \cap b\alpha^{-1} \neq \varnothing$. Hence $a, b \in C\alpha$. This is a contradiction. Hence $(\alpha)_{\mathrm{q}} \neq (\alpha)_{\mathrm{b}}$. By Proposition 3, we conclude that $T(X,\sigma,\rho)$ is not a BQ-semigroup.

Conversely, assume that the converse conditions hold. If $\sigma = I_X$ or $\rho = X \times X$, then by Proposition 1, we have $T(X,\sigma,\rho) = T(X)$ is a regular semigroup. Thus $T(X,\sigma,\rho)$ is a BQ-semigroup. If $\rho = I_X$, then $T(X,\sigma,\rho)$ is a BQ-semigroup by Corollary 1.

Suppose that $\sigma = X \times X$. Let α, $\beta \in T(X,\sigma,\rho)$. If $\alpha = \beta$, then by Corollary 2 we have $(\alpha)_{\mathrm{b}} = (\alpha)_{\mathrm{q}}$. Assume that $\alpha \neq \beta$. Let $\gamma \in (\alpha,\beta)_{\mathrm{q}}$. We consider four cases as follows.

Case 1: $\gamma \in \alpha T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\alpha$. Then by Proposition 2, we have $\gamma \in (\alpha)_{\mathrm{q}}$. Since $\sigma = X \times X$, $\gamma \in (\alpha)_{\mathrm{q}} = (\alpha)_{\mathrm{b}}$ by Corollary 2. By minimality of $(\alpha)_{\mathrm{b}}$, we deduce that $\gamma \in (\alpha,\beta)_{\mathrm{b}}$.

Case 2: $\gamma \in \beta T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\beta$. Then $\gamma \in (\beta)_{\mathrm{q}}$. Since $\sigma = X \times X$, $\gamma \in (\beta)_{\mathrm{q}} = (\beta)_{\mathrm{b}}$ by Corollary 2. It follows that $\gamma \in (\alpha,\beta)_{\mathrm{b}}$.

Case 3: $\gamma \in \alpha T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\beta$. Then $\gamma = \alpha\alpha' = \beta'\beta$ for some α', $\beta' \in T(X,\sigma,\rho)$. For each $y \in X\alpha$, we choose and fix $a_y \in X$ such that $a_y\alpha = y$. Define $\mu : X \to X$ by

$$x\mu = \begin{cases} a_x\beta', & x \in X\alpha, \\ x\beta', & \text{otherwise}. \end{cases}$$

Since $\sigma = X \times X$ and $\beta' \in T(X,\sigma,\rho)$, we have that $\mu \in T(X,\sigma,\rho)$. Let $x \in X$. Since $\gamma = \alpha\alpha' = \beta'\beta$, we deduce that

$$x\alpha\mu\beta = a_{x\alpha}\beta'\beta = a_{x\alpha}\alpha\alpha' = x\alpha\alpha' = x\gamma.$$

Hence $\gamma = \alpha\mu\beta \in \alpha T(X,\sigma,\rho)\beta \subseteq (\alpha,\beta)_{\mathrm{b}}$.

Case 4: $\gamma \in \beta T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\alpha$. Then $\gamma = \alpha'\alpha = \beta\beta'$ for some α', $\beta' \in T(X,\sigma,\rho)$. For each $y \in X\beta$, we choose and fix $a_y \in X$ such that $a_y\beta = y$. Define $\mu : X \to X$ by

$$x\mu = \begin{cases} a_x\alpha', & x \in X\alpha, \\ x\alpha', & \text{otherwise}. \end{cases}$$

Since $X/\sigma = \{X\}$ and $\alpha' \in T(X,\sigma,\rho)$, we deduce that $\mu \in T(X,\sigma,\rho)$. Let $x \in X$. Since $\gamma = \alpha'\alpha = \beta\beta'$, we obtain that

$$x\beta\mu\alpha = a_{x\beta}\alpha'\alpha = a_{x\beta}\beta\beta' = x\beta\beta' = x\gamma.$$

Then $\gamma = \beta\mu\alpha \in \beta T(X,\sigma,\rho)\alpha \subseteq (\alpha,\beta)_{\mathrm{b}}$.

We deduce that $(\alpha,\beta)_{\mathrm{b}} = (\alpha,\beta)_{\mathrm{q}}$. It follows from Proposition 3 that $T(X,\sigma,\rho)$ is a BQ-semigroup. □

Next, we show that the semigroup $T(X,\sigma,\rho)$ can be embeddable in $T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$.

Theorem 2 *$T(X,\sigma,\rho)$ can be embeddable in $T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$.*

Proof: Let $Y=\sigma$ and $Z=\rho$. Then $Z\subseteq Y$. For each $\alpha\in T(X,\sigma,\rho)$, we define $\beta_\alpha\in T(Y)$ by

$$(x,y)\beta_\alpha=(x\alpha,y\alpha) \text{ for all } (x,y)\in Y.$$

Since $\alpha\in T(X,\sigma,\rho)$, it follows that $Y\beta_\alpha\subseteq Z$. Hence β_α is well defined. Define $\phi: T(X,\sigma,\rho)\to T(Y,Z)$ by

$$\alpha\phi=\beta_\alpha \text{ for all } \alpha\in T(X,\sigma,\rho).$$

Let α_1, $\alpha_2\in T(X,\sigma,\rho)$ be such that $\alpha_1\phi=\alpha_2\phi$. Then $\beta_{\alpha_1}=\beta_{\alpha_2}$. If $x\in X$ then $(x,x)\in Y$ and

$$(x\alpha_1,x\alpha_1)=(x,x)\beta_{\alpha_1}=(x,x)\beta_{\alpha_2}=(x\alpha_2,x\alpha_2).$$

Hence $x\alpha_1=x\alpha_2$ for all $x\in X$ and so $\alpha_1=\alpha_2$. This shows that ϕ is injective. Next we claim that $\beta_{\alpha_1\alpha_2}=\beta_{\alpha_1}\beta_{\alpha_2}$. If $(x,y)\in Y$ then

$$\begin{aligned}(x,y)\beta_{\alpha_1\alpha_2}&=(x\alpha_1\alpha_2,y\alpha_1\alpha_2)\\&=(x\alpha_1,y\alpha_1)\beta_{\alpha_2}\\&=(x,y)\beta_{\alpha_1}\beta_{\alpha_2},\end{aligned}$$

as required. □

Theorem 3 *Let $\varphi: S\to T$ be a semigroup isomorphism. If S is a BQ-semigroup, then T is also a BQ-semigroup.*

Nenthein and Kemprasit[11] proved that $T(X,Y)$ is a *BQ*-semigroup. As a consequence of Theorem 3, the following result follows readily.

Corollary 3 *If $T(X,\sigma,\rho)\cong T(Y,Z)$ for some sets Y, Z with $Z\subseteq Y$, then $T(X,\sigma,\rho)$ is a BQ-semigroup.*

The following result follows immediately from Corollary 3 and Theorem 1.

Corollary 4 *If $T(X,\sigma,\rho)\cong T(Y,Z)$ for some sets Y, Z with $Z\subseteq Y$, then $\sigma=X\times X$ or $\sigma=I_X$ or $\rho=X\times X$ or $\rho=I_X$.*

Finally, we give the necessary conditions for the semigroups $T(X,\sigma,\rho)$ and $T(Y,Z)$ to be isomorphic. In what follows, $|A|$ means the cardinality of a set A.

Theorem 4 (Ref. 7) *$T(X)\cong T(Y)$ if and only if $|X|=|Y|$.*

Proposition 5 *If $\sigma=I_X$ or $\rho=X\times X$, then $T(X,\sigma,\rho)\cong T(Y,Z)$ for some sets Y, Z with $Z\subseteq Y$.*

Proof: Suppose that $\sigma=I_X$ or $\rho=X\times X$.

Case 1: $\sigma=I_X$. Then $\sigma=\rho$. By Proposition 1, we obtain $T(X,\sigma,\rho)=T(X)$. Let $Y=Z=\sigma$. Then $T(Y,Z)=T(Y)$. Since $\sigma=I_X$ we deduce that $|X|=|I_X|=|\sigma|=|Y|$. This implies that $T(X)\cong T(Y)$ by Theorem 4.

Case 2: $\rho=X\times X$. Then $\sigma=\rho$. Thus $T(X,\sigma,\rho)=T(X)$. Let $Y=Z=I_X$. Then $T(Y,Z)=T(Y)$. Since $Y=I_X$ it follows that $|X|=|I_X|=|Y|$. Hence $T(X)\cong T(Y)$. □

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REFERENCES

1. Doss C (1995) Certain equivalence relations in transformation semigroups. MSc thesis, Univ of Tennessee, Knoxville.
2. Pei H (2005) Regularity and Green's relations for semigroups of transformations that preserve an equivalence. *Comm Algebra* **33**, 109–18.
3. Sullivan RP, Mendes-Gonçalves S (2010) Semigroups of transformations restricted by an equivalence. *Cent Eur J Math* **8**, 1120–31.
4. Lajos S (1961) Generalized ideals in semigroups. *Acta Sci Math* **22**, 217–22.
5. Kapp KM (1969) On bi-ideals and quasi-ideals in semigroups. *Publ Math Debrecen* **16**, 179–85.
6. Steinfeld O (1978) *Quasi-ideals in Rings and Semigroups*, Akadémiai Kiadó Budapest.
7. Clifford AH, Preston GB (1961) *The Algebraic Theory of Semigroups*, Vol I, Mathematical Surveys, American Mathematical Society, Rhode Island.
8. Calais J (1968) Demigroupes dans lesquels tout bi-idéal est un quasi-idéal. In: *Semigroup Symposium*, Smolenice.
9. Symons JSV (1975) Some results concerning a transformation semigroup. *J Aust Math Soc* **19**, 413–25.
10. Nenthein S, Youngkhong P, Kemprasit Y (2005) Regular elements of some transformation semigroups. *Pure Math Appl* **16**, 307–14.
11. Nenthein S, Kemprasit Y (2006) On transformation semigroups which are *BQ*-semigroups. *Int J Math Math Sci* **2006**, 1–10.
12. Sanwong J, Sommanee W (2008) Regularity and Green's relations on a semigroup of transformations with restricted range. *Int J Math Math Sci* **2008**, 1–11.