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OVD-characterization of simple K_3-groups

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ABSTRACT: A vanishing element of G is an element $g\in G$ such that $\chi(g)=0$ for some $\chi\in\mathrm{Irr}(G)$. Let $\mathrm{Van}(G)$ denote the set of vanishing elements of G, i.e., $\mathrm{Van}(G)=\{g\in G\mid \chi(g)=0 \text{ for some } \chi\in\mathrm{Irr}(G)\}$. We define $\mathrm{vo}(G)$ to be the set $\{o(g)\mid g\in\mathrm{Van}(G)\}$ consisting of the orders of the elements in $\mathrm{Van}(G)$, that is, $\mathrm{vo}(G)=\{o(g)\mid g\in\mathrm{Van}(G)\}$. Obviously, $vo(G)\subseteq\omega(G)$ where $\omega(G)$ is the set of element orders of G. Let $\pi_V(G)$ be the set of prime divisors of the orders of the vanishing elements of G, that is, $\pi_V(G)=\{\pi(o(g))\mid g\in\mathrm{Van}(G)\}$. Obviously $\pi_V(G)\subseteq\pi(G)$ where $\pi(G)$ denotes the set of the prime divisors of the order $|G|$ of a group G. Let G be a finite group and $|G|=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}p_{k+1}^{\alpha_{k+1}}\cdots p_n^{\alpha_n}$, where the p_i are different primes and the α_i are positive integers. Assume that $\pi_V(G)=\{p_1,p_2,\cdots,p_k\}$. For $p\in\pi_V(G)$, let $\deg(p):=|\{q\in\pi_V(G)|p\sim q\}|$, which we call the vanishing degree of p. We also define $VD(G):=(\deg(p_1),\deg(p_2),\cdots,\deg(p_k))$, where $p_1<p_2<\cdots<p_k$. We call $VD(G)$ the vanishing degree pattern of G. In this paper, we give a characterization of simple K_3-groups by group orders and their vanishing degree patterns of the vanishing prime graphs.

KEYWORDS: element order, alternating group, degree pattern, vanishing prime graph, simple group

MSC2010: 20C15 20C33

INTRODUCTION

All groups in this paper are finite, and for a simple group, it is non-abelian. Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of a group G. Let $\mathrm{cd}(G)$ be the set of degrees of all irreducible complex characters of G. A vanishing element of G is an element $g\in G$ such that $\chi(g)=0$ for some $\chi\in\mathrm{Irr}(G)$. Let $\mathrm{Van}(G)$ denote the set of vanishing elements of G, i.e., $\mathrm{Van}(G)=\{g\in G\mid\chi(g)=0 \text{ for some } \chi\in\mathrm{Irr}(G)\}$. We define $\mathrm{vo}(G)$ to be the set $\{o(g)\mid g\in\mathrm{Van}(G)\}$ consisting of the orders of the elements in $\mathrm{Van}(G)$, i.e.,

$$\mathrm{vo}(G)=\{o(g)\mid g\in\mathrm{Van}(G)\}.$$

Clearly, $\mathrm{vo}(G)\subseteq\omega(G)$ where $\omega(G)$ is the set of element orders of G. Let $\pi_{\mathrm{v}}(G)$ be the set of prime divisors of the orders of the vanishing elements of G, that is,

$$\pi_{\mathrm{v}}(G)=\{\pi(o(g))\mid g\in\mathrm{Van}(G)\}.$$

Clearly $\pi_{\mathrm{v}}(G)\subseteq\pi(G)$ where $\pi(G)$ denotes the set of the prime divisors of the order $|G|$ of a group G. Now the *vanishing prime graph* of G, denoted by $\Gamma(G)$, is the graph whose vertices are the prime divisors of the orders of the elements in $\mathrm{Van}(G)$, and two distinct vertices p and q are adjacent, denoted by $p\sim q$, if and only if $\mathrm{Van}(G)$ contains an element of order divisible by $p\cdot q$ [1].

As in Ref. 2, we define the following concepts.

Definition 1 Let G be a finite group and $|G|=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}p_{k+1}^{\alpha_{k+1}}\cdots p_n^{\alpha_n}$, where the p_i are different primes and the α_i are positive integers. Assume that $\pi_{\mathrm{v}}(G)=\{p_1,p_2,\dots,p_k\}$. For $p\in\pi_{\mathrm{v}}(G)$, let

$$\deg(p):=|\{q\in\pi_{\mathrm{v}}(G)\mid p\sim q\}|,$$

which we call the vanishing degree of p. We also define

$$\mathrm{VD}(G):=(\deg(p_1),\deg(p_2),\dots,\deg(p_k)),$$

where $p_1<p_2<\cdots<p_k$. We call $\mathrm{VD}(G)$ the *vanishing degree pattern* of G or just the *degree pattern*.

Clearly, if $\deg(p)=0$, then there is a character $\chi\in\mathrm{Irr}(G)$ and an element $g\in G$ of order p-power of G such that $\chi(g)=0$ and p is a connected component in $\Gamma(G)$.

Generally, degree pattern cannot determine the structure of G. For instance, the symmetric group of degree 3, and the quaternion 2-group of order 8 have the same degree pattern, although they are not isomorphic.

Let G and M be finite groups satisfying the conditions:
(C_1) $|G| = |M|$, and
(C_2) $VD(G) = VD(M)$.
Then we have the following questions:
(i) What is the influence of these conditions on the structure of G?
(ii) Is the number of the non-isomorphic groups enjoying the conditions C_1 and C_2 finite?

We find that D_8, the dihedral group, and Q_8, the quaternion group (which have the same order), have the same degree pattern, but they are not isomorphic.

Definition 2 A group M is said to be k-fold OVD-characterizable if there are exactly k non-isomorphic groups with the properties C_1 and C_2. Furthermore, a 1-fold OVD-characterizable group is simply called an OVD-characterizable group.

Here we give a characterization of simple K_3-groups by their degree patterns and orders (a group G is called a simple K_n-group if G is simple and $|\pi(G)| = n$).

Theorem 1 *Simple K_3-groups are OVD-characterizable.*

We introduce some notation here. Let G be a group and r be a prime divisor of $|G|$. Then denote the set of Sylow r-subgroups G_r of G by $\mathrm{Syl}_r(G)$. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ denote the automorphism and outer-automorphism groups of G, respectively. Let A_n be the alternating group of degree n. Let $L_n(q)$ be a projective special linear group of degree n over a finite field of order q and $U_n(q)$ a projective unitary group of degree n over a finite field of order q. All other symbols are standard[3].

SOME PRELIMINARY RESULTS

Lemma 1 *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Proof: See Lemma 1 of Ref. 4. □

Lemma 2 *Let G be a finite group and let $\pi(G)$ ($\pi_v(G)$) be the set of the prime divisors of (vanishing) elements of G. Then $\pi_v(G) \subseteq \pi(G)$. If G is a non-abelian simple group, then $\pi_v(G) = \pi(G)$.*

Proof: Clearly, an element of G need not be a vanishing element of G, and so $\pi_v(G) \subseteq \pi(G)$. Furthermore if G is a non-abelian simple group, then $\pi_v(G) = \pi(G)$ by Proposition 6.4 of Ref. 5 and p. 10 of Ref. 3. □

Lemma 3 *Let $G = A \times B$. Let $p \in \pi_v(A)$ and $q \in \pi_v(B)$. Then $pq \in \mathrm{vo}(G)$. Furthermore, if $n \in \omega(A)$, then $nq \in \mathrm{vo}(G)$.*

Proof: By hypotheses, there is a character $\psi \in \mathrm{Irr}(A)$ such that $\chi(a) = 0$ for some $a \in A$ with $o(a) = p$. Also, there exists a character $\theta \in \mathrm{Irr}(B)$ such that $\theta(b) = 0$ for some $b \in B$ with $o(b) = q$. Clearly there is an element $g \in G$ such that $g = ab$. Then by Theorem 4.21 of Ref. 6, $\psi\theta \in \mathrm{Irr}(G)$. Now, $(\psi\theta)(g) = \psi(a)\theta(b) = 0$ and so there is a vanishing element of order pq. Furthermore, if $n \in \omega(A)$ and $q \in \pi_v(B)$, then there exists a character $\theta \in \mathrm{Irr}(B)$ such that $\theta(b) = 0$ for some $b \in B$ with $o(b) = q$ and hence we have $(\psi\theta)(g) = \psi(a)\theta(b) = \psi(a) \cdot 0 = 0$. Thus $nq \in \mathrm{vo}(G)$. □

PROOF OF THEOREM 1

Let $n_r = r^a$ or $r^a \| n$ be the r-part of the positive integer n where a is a positive integer such that $r^a \mid n$ but $r^{a+1} \nmid n$. If H is characteristic in G, then we write $H \,\mathrm{ch}\, G$. If a positive integer n divides a positive integer m, then we write $n \mid m$.

Proof: Let M be a simple K_3-group and G a group. Hence $VD(G) = VD(M)$ and $|G| = |M|$.

By Lemma 2, $\pi_v(G) \subseteq \pi(G)$. So in the proof of Theorem 1, we first show $\pi_v(G) = \pi(G)$, then prove that G is non-solvable, and finally obtain the desired result.

Case 1: $M \in \{A_5, A_6, L_2(7), L_2(8), L_2(17)\}$. By Lemma 2 and the structure of $VD(G) = VD(M)$, we have $\pi_v(G) = \pi(G)$. In this case, $VD(G) = (0, 0, 0)$ and so $\Gamma(G)$ has three connected components. Assume that G is solvable. Then by Theorem A of Ref. 1, $\Gamma(G)$ has at most two connected components, a contradiction. Hence G is non-solvable. Now Lemma 1 results that G has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that K/H is a product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Then we consider this case by case.

Case 1.1: $M = A_5$. Then K/H is isomorphic to A_5 and $|G/K| \mid 2$. If $|G : K| = 1$, then order consideration results that $H = 1$ and so G is isomorphic to A_5. If $|G/K| = 2$, then $|G : K|_2 \cdot |K : H|_2 > |G|_2$, a contradiction.

Case 1.2: $M = A_6$. Now $K/H \cong A_5$ or A_6. If $K/H \cong A_5$, then $|G/K| \mid 2$. If $|G/K| = 1$, then $H = 6$.

Note that the groups of order 6 are S_3, and Z_6 where S_n is the symmetric group of degree n, and Z_n is the cyclic group of order n. We obtain G is isomorphic to $A_5 \times Z_6, A_5 \times S_3$ or $(2.A_5) \times Z_3$. Lemma 3 forces $2 \cdot 5 \in \mathrm{vo}(G)$ and so $2 \sim 5$, a contradiction to $\deg(5) = 0$. If $|G/K| = 2$, then $|H| = 3$. Now G is isomorphic to $Z_3 \times S_5$ and so $3 \cdot 5 \in \mathrm{vo}(G)$. It follows that $3 \sim 5$, a contradiction.

If $K/H \cong A_6$, then $H = 1$ and so order comparison implies that G is isomorphic to A_6.1.3 $M = L_2(7)$. Now, $K/H \cong L_2(7)$. Then $H = 1$ and order consideration forces $G \cong L_2(7)$.1.4 $M = L_2(8)$. Now we have that $K/H \cong L_2(7)$ or $L_2(8)$.

If $K/H \cong L_2(7)$, then $|G/K| \mid 2$. If $|G/K| = 1$, then $|H| = 6$ and so G is isomorphic to one of the groups $L_2(7) \times Z_6$, $L_2(7) \times S_6$ or $2.L_2(7) \times Z_3$. By Lemma 3, we can rule out this case since $3 \sim 7$. If $|G/K| = 2$, then G is isomorphic to $SL_2(7) \times Z_3$ and so $3 \sim 7$, which is also a contradiction since $3 \sim 7$.

If $K/H \cong L_2(8)$, then $H = 1$ and so $G \cong L_2(8)$ by considering the group order.

Case 1.5: $M = L_2(17)$. Now we have $K/H \cong L_2(17)$ and so $H = 1$. By order comparison $G \cong L_2(17)$.

Case 2: $M \in \{U_3(3), L_3(3), U_4(2)\}$. In this case, $\mathrm{VD}(G) = (1, 1, 0)$. We consider the following steps

Step 1: $\pi_{\mathrm{v}}(G) = \pi(M)$. Since the proof is similar, we consider only one case, for instance $M = U_3(3)$. Then $l(M) = 1$, $\mathrm{Out}(M) = 2$, where $l(M)$ is the Schur multiplier of M and $\mathrm{Out}(M)$ is an outer-automorphism group of G, and $\pi(M) = \{2, 3, 7\}$.

Suppose that $7 \notin \pi_{\mathrm{v}}(G)$. Then there is no character $\chi \in \mathrm{Irr}(G)$ and an element $g \in G$ with order 7 such that $\chi(g) = 0$. Let P be a Sylow 7-subgroup of G. Then by Theorem C of Ref. 5 the following conclusions hold: (a) P is normal in G; (b) either G is abelian, or $G/O_{p'}(G)$ is a Frobenius group with kernel $PO_{p'}(G)/O_{p'}(G)$, and $O_{p'}(G)$ is nilpotent. If P is normal in G and G is abelian, then we obtain a contradiction to $\deg(2) = 0$. Thus P is normal in G and $G/O_{p'}(G)$ has kernel $PO_{p'}(G)/O_{p'}(G)$. Note that in this case G is non-abelian and so there is always a vanishing element in G. It follows that $|G/O_{p'}(G) : PO_{p'}(G)/O_{p'(G)}| \mid (|PO_{p'}(G)/O_{p'}(G)| - 1)$, namely, $|G : PO_{p'}(G)| \mid (|P| - 1)(= 6)$. Then we have four cases: $G = PO_{p'}(G)$, $|G| = 2|PO_{p'}(G)|$, $|G| = 3|PO_{p'}(G)|$, and $|G| = 6|PO_{p'}(G)|$.

If $G = PO_{p'}(G)$, then G is nilpotent and so if there is a vanishing element of G, Lemma 3 implies that $\deg(7) \neq 0$, a contradiction. Hence that G is abelian is not the case.

If $|G| = 2|PO_{p'}(G)|$, then $O_{p'}(G)$ is nilpotent and $O_{\{2,3\}}(G) = O_{p'}(G)$. Let Q and R be a Sylow 3-subgroup and 2-subgroup of $O_{p'}(G)$. Then the nilpotence of $O_{p'}(G)$ results in Q and R being normal in $O_{p'}(G)$ and so Q, $R \,\mathrm{ch}\, O_{p'}(G)$. Then if there is always a vanishing element, we have $\deg(7) \neq 0$ by Lemma 3, a contradiction. Similarly, we can rule out the cases $|G| = 3|PO_{p'}(G)|$ and $|G| = 6|PO_{p'}(G)|$. Hence $7 \in \pi_{\mathrm{v}}(G)$. Similarly we can obtain $2, 3 \in \pi_{\mathrm{v}}(G)$.

Step 2: G is non-solvable. Also in this step, we say $M = U_3(3)$ since the proof is similar. Assume that G is solvable. Let N be a minimal normal subgroup. Then N is an elementary abelian p-group. If $p = 7$, then $N = G_7$, the Sylow 7-subgroup of G, is normal in G and so by Lemma 2.1(a) of Ref. 5, $xG_7 \subseteq \mathrm{Van}(G)$ with $|x| = 2 \in \pi_{\mathrm{v}}(G)$ by Step 1, that is, there is an element of order $2{\cdot}7$, a contradiction to $\deg(7) = 0$. If N is 2-group or 3-group, we can also obtain a contradiction since $7 \in \pi_{\mathrm{v}}(G)$ by Step 1. Thus G is non-solvable.

Step 3: G is isomorphic to M. By Step 2, G is non-solvable. Now Lemma 1 shows that G has a normal series $1 \lhd H \lhd K \lhd G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.

We need to consider the cases $M \in \{U_3(3), L_3(3), U_4(2)\}$.

Case 3.1: $M = U_3(3)$. Then K/H is isomorphic to one of the groups $L_2(7)$, $L_2(8)$, or $U_3(3)$. If $K/H \cong L_2(7)$, then $|G : K| \mid 2$. If $|G/K| = 1$, then G is isomorphic to $H \times L_2(7)$ with $|H| = 36$ or $W \times 2.L_2(7)$ with $|W| = 18$. If $|G/K| = 2$, then G is isomorphic to $H \times 2.L_2(7)$ with $|H| = 18$ or $W \times 2.L_2(7).2$ with $|W| = 9$. So by Lemma 3, we have either $\deg(7) = 1$ or $\deg(7) = 2$, a contradiction to the hypotheses.

If $K/H \cong L_2(8)$, then $|G/K| \mid 3$. If $|G/K| = 1$, then $G \cong H \times L_2(8)$ with $|H| = 12$. By Lemma 3, $3 \cdot 7 \in \mathrm{vo}(G)$, a contradiction to $\deg(7) = 0$. If $|G/K| = 3$, then $G \cong H \times SL_2(8)$ with $|H| = 4$, and so we obtain $2 \sim 7$, a contradiction.

If $K/H \cong U_3(3)$, then $H = 1$ and so order comparison results in $G \cong U_3(3)$.

Case 3.2: $M = L_3(3)$. In this case, $K/H \cong L_3(3)$ and so $H = 1$. Order consideration proves that $G \cong L_3(3)$.

Case 3.3: $M = U_4(2)$. Then K/H is isomorphic to one of the groups A_5, A_6, and $U_4(2)$. If $K/H \cong A_5$, then $|G/K| \mid 2$. If $|G/K| = 1$, then G has one of the structures $H \times A_5$ with $|H| = 2^4 \cdot 3^3$ and $H_1 \times 2.A_5$ with $|H_1| = 2^3 \cdot 3^3$. If $|G/K| = 2$, then G is isomorphic to one of the groups $H \times S_5$ with $|H| = 2^3 \cdot 3^3$, $H_1 \times 2.A_5.2$ with $|H_1| = 2^2 \cdot 3^3$. So by Lemma 3, we have $\deg(5) \neq 0$, a contradiction to the hypotheses.

If $K/H \cong A_6$, then $l(A_6) = 6$ and $|\mathrm{Out}(A_6)| = 2^2$.

If $|G/K| = 1$, then G is isomorphic to $W_1 \times k_1.L_2(9)$ with $k_1 \mid 6$. If $|G/K| = 2$, then G is isomorphic to $W_2 \times k_2.L_2(9).2_{k_3}$ with $k_2 \mid 12$ and $k_3 \in \{1,2,3\}$. If $|G/K| = 4$, then G is isomorphic to $W_3 \times k_4.SL_2(9)$ with $k_4 \mid 6$. In these cases, Lemma 3 implies that $2 \sim 5$, a contradiction to $\deg(5) = 0$ since $W_i \neq 1$ with $i \in \{1,2,3\}$.

If $K/H \cong U_4(2)$, then $|H| = 1$ and so order consideration implies $G \cong U_4(2)$. □

SOME APPLICATIONS

Conjecture 1 (Ref. 7) *Let G be a finite group and let M be a finite non-abelian simple group. If* $\mathrm{vo}(G) = \mathrm{vo}(M)$ *and* $|G| = |M|$, *then* $G \cong M$.

Corollary 1 *Let G be a finite group and let M be a non-abelian simple* K_3*-group. If* $\mathrm{vo}(G) = \mathrm{vo}(M)$ *and* $|G| = |M|$, *then* $G \cong M$.

Proof: By Ref. 3, we have $\mathrm{vo}(G) = \mathrm{vo}(M)$ and so $\mathrm{VD}(G) = \mathrm{VD}(M)$. Then by Theorem 1, $G \cong M$. Also, we can obtain the result from Ref. 7. □

Corollary 2 *Let M be a finite non-abelian simple* K_3*-group. Then M is OD-characterizable.*

Proof: By Ref. 3, $\mathrm{vo}(G) = \mathrm{vo}(M)$ and so $\mathrm{VD}(G) = \mathrm{VD}(M) = D(G) = D(M)$. By Theorem 1, $G \cong M$. □

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REFERENCES

1. Dolfi S, Pacifici E, Sanus L, Spiga P (2010) On the vanishing prime graph of solvable groups. *J Group Theor* **13**, 189–206.
2. Moghaddamfar AR, Zokayi AR, Darafsheh MR (2005) A characterization of finite simple groups by the degrees of vertices of their prime graphs. *Algebra Colloq* **12**, 431–42.
3. Conway JH, Curtis RT, Norton SP, Parker RA, Wilson RA (1985) *Atlas of Finite Groups*, Oxford Univ Press.
4. Xu H, Chen G, Yan Y (2014) A new characterization of simple K_3-groups by their orders and large degrees of their irreducible characters. *Comm Algebra* **42**, 5374–80.
5. Dolfi S, Pacifici E, Sanus L, Spiga P (2010) On the vanishing prime graph of finite groups. *J Lond Math Soc 2* **82**, 167–83.
6. Isaacs IM (1994) *Character Theory of Finite Groups*, Dover Publications, Inc., New York, corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)].
7. Gasemabadi MF, Iranmanesh A, Mavadatpur F (2015) A new characterization of some finite simple groups. *Sibirsk Mat Zh* **56**, 94–9.