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The unique positive solution for fractional integro-differential equations on infinite intervals

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ABSTRACT: By means of some properties of normal cones and a recent fixed point theorem for monotone operators, we establish the existence and uniqueness of positive solutions for fractional integro-differential equations on infinite intervals. Our results complement the previous results of positive solutions for fractional integro-differential equations. Furthermore, we can make an iterative scheme to approximate the unique positive solution. An example is given to demonstrate our main result.

KEYWORDS: fixed point theorem; normal cone

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INTRODUCTION

In recent decades, fractional differential equations (FDEs) have been used to describe many phenomena in different fields of engineering and scientific disciplines which include physics, chemistry, biology, biophysics, economics, signal and image processing, control theory, aerodynamics, electromagnetism, and viscoelasticity[1–4]. The existence of solutions of FDEs with boundary conditions has been studied widely[5–8]. Most papers considered the existence of positive solutions[9–12]. Recently, more and more attention has been paid to fractional integro-differential equations (FIDEs) with boundary conditions[13–16]. The methods used there are mainly the Banach fixed point theorem, the Leray-Schauder theorem, and fixed point theorems in cones[16–19].

In Ref. 19, the authors considered the following FIDE with boundary conditions:

$$\begin{gathered} D^{\alpha}u(t)+f(t,u(t),Tu(t),Su(t))=\theta,\\ n-1<\alpha\leqslant n,\ n\in\mathbb{N},\ n\geqslant 2,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=\theta,\\ D^{\alpha-1}u(\infty)=u_{\infty}, \end{gathered}$$

where $t\in J:=[0,\infty)$, $f\in C[J\times E\times E\times E,E]$, $u_{\infty}\in E$, θ is the zero element in E, and D^{α} is the Riemann-Liouville fractional derivative,

$$\begin{aligned} (Tu)(t)&=\int_0^t k(t,s)u(s)\,\mathrm{d}s,\\ (Su)(t)&=\int_0^{\infty} h(t,s)u(s)\,\mathrm{d}s, \end{aligned}$$

with $k(t,s)\in C[D,\mathbb{R}]$, $h(t,s)\in C[D_0,\mathbb{R}]$, $D=\{(t,s)\in\mathbb{R}^2 \mid 0\leqslant s\leqslant t\}$, $D_0=J\times J$. The existence and uniqueness of solutions was obtained by using the Banach fixed point theorem and the monotone iterative technique.

In Ref. 18, the authors concentrated on the following FIDE with integral boundary conditions:

$$\begin{gathered} D_{\mathrm{C}}^{\alpha}u(t)+f(t,u(t),Tu(t),Su(t))=0,\ 0<t<1,\\ u(0)=b_0,\ u'(0)=b_1,\ \cdots,\ u^{(n-3)}(0)=b_{n-3},\\ u^{(n-1)}(0)=b_{n-1},\qquad u(1)=\mu\int_0^1 u(s)\,\mathrm{d}s, \end{gathered}$$

where $n-1<\alpha\leqslant n$, $0\leqslant\mu<n-1$, $n\geqslant 3$, $b_i\geqslant 0$ $(i=1,2,\ldots,n-3,n-1)$, and D_{C}^{α} denotes the Caputo fractional derivative. The authors obtained the uniqueness and existence of positive solutions using the Krasnosel'skii fixed point theorem and the Banach contraction principle.

Recently, Henderson and Luca[15] gave the existence of nonnegative solutions for the multi-point

FIDE

$$\begin{gathered}D^{\alpha}u(t)+f(t,u(t),Tu(t),Su(t))=0,\ t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ D^{p}u(t)|_{t=1}=\sum_{i=1}^{\mathrm{m}}a_iD^{q}u(t)|_{t=\xi_i},\end{gathered}$$

where $n-1<\alpha\leqslant n$, $n\geqslant 3$, $a_i\in\mathbb{R}$, $0<\xi_1<\cdots<\xi_{\mathrm{m}}\leqslant 1$, $p\in[1,n-2]$, and $q\in[0,p]$,

The main approaches are the Krasnosel'skii fixed point theorem for the sum of two operators and the Banach contraction mapping principle. However, we found that there are still very few papers reported on the uniqueness of positive solutions for integro-differential fractional boundary value problems on infinite intervals. The purpose of this paper is to discuss the existence and uniqueness of positive solutions for FIDEs with boundary conditions on infinite intervals. Motivated by the results in Refs. 20–22, we investigate the following FIDE with boundary conditions:

$$\left.\begin{gathered}D^{\alpha}u(t)+a(t)f(t,u(t),(Tu)(t))=0,\\ n-1<\alpha\leqslant n,\ n\in\mathbb{N},\ n\geqslant 2,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0,\\ D^{\alpha-1}u(\infty)=\beta,\end{gathered}\right\}\quad(1)$$

where $t\in J:=[0,\infty)$, $a\in C(J,J)$, $f\in C[J\times J\times J,J]$, $\beta>0$, $(Tu)(t)=\int_0^t k(t,s)u(s)\,\mathrm{d}s$ with $k(t,s)\in C[D,J]$, $D=\{(t,s)\in\mathbb{R}^2\mid 0\leqslant s\leqslant t\}$. We will give the existence and uniqueness of positive solutions for (1) and use iteration to approximate the unique positive solution. The methods used here are some properties of normal cones and a recent fixed point theorem for monotone operators. We can obtain the existence and uniqueness of positive solutions for the following problem:

$$\left.\begin{gathered}D^{\alpha}u(t)+a(t)f(t,u(t),(Tu)(t),Su(t))=0,\\ n-1<\alpha\leqslant n,\ n\in\mathbb{N},\ n\geqslant 2,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0,\\ D^{\alpha-1}u(\infty)=\beta,\end{gathered}\right\}\quad(2)$$

where $t\in J$, $a\in C(J,J)$, $f\in C[J\times J\times J\times J,J]$, $\beta>0$, T is as in (1), $(Su)(t)=\int_0^{\infty}h(t,s)u(s)\,\mathrm{d}s$ with $h(t,s)\in C[J\times J,J]$.

PRELIMINARIES

Throughout this paper, we suppose that the following conditions hold.

(H_1) $f\in C[J\times J\times J,J]$ and $f(t,u,v)$ are increasing with respect to the second and third variables. When u, v are bounded, $f(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})v)$ is bounded for $t\in J$.

(H_2) $k(t,s)\geqslant 0$ and $k^*:=\sup_{t\in J}\int_0^t k(t,s)\,\mathrm{d}s<\infty$.

(H_3) $a\in C(J,J)$ and $0<\int_0^{\infty}a(s)\,\mathrm{d}s<\infty$.

(H_4) For $r\in(0,1)$, there exists $\varphi(r)\in(r,1)$ such that $f(t,ru,rv)\geqslant\varphi(r)f(t,u,v)$, $\forall t,u,v\in J$.

Definition 1 [Refs. 5, 12] Let $f:[0,\infty)\to\mathbb{R}$ be continuous. The Riemann-Liouville fractional derivative of order δ for f is given by

$$D^{\delta}f(t)=\frac{1}{\Gamma(n-\delta)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t(t-s)^{n-\delta-1}f(s)\,\mathrm{d}s,$$

$n=[\delta]+1$ where $[\cdot]$ denotes the integer part.

Definition 2 [Refs. 5, 12] Let $f:[0,\infty)\to\mathbb{R}$ be continuous. The Riemann-Liouville fractional integral of order δ for f is given by

$$I^{\delta}f(t)=\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}f(s)\,\mathrm{d}s,\ \delta>0.$$

We will use the following space $FC(J,\mathbb{R})$ to study (1):

$$FC(J,\mathbb{R})=\left\{u\in C[J,\mathbb{R}]:\ \sup_{0\leqslant t<\infty}\frac{|u(t)|}{1+t^{\alpha-1}}<\infty\right\}.$$

From Ref. 20, we know that $FC(J,\mathbb{R})$ is a Banach space equipped with the norm

$$\|u\|=\sup_{0\leqslant t<\infty}\frac{|u(t)|}{1+t^{\alpha-1}}<\infty.$$

Lemma 1 *Let* (H_3) *be satisfied and* $y\in FC(J,\mathbb{R})$ *with* $y(t)$ *bounded on* $[0,\infty)$. *Then*

$$\left.\begin{gathered}D^{\alpha}u(t)+a(t)y(t)=0,\\ n-1<\alpha\leqslant n,\ n\in\mathbb{N},\ n\geqslant 2,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0,\\ D^{\alpha-1}u(\infty)=\beta,\end{gathered}\right\}\quad(3)$$

has a unique solution

$$u(t)=\int_0^{\infty}G(t,s)a(s)y(s)\,\mathrm{d}s+\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)},\quad(4)$$

where

$$\Gamma(\alpha)G(t,s)=\begin{cases}t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leqslant s\leqslant t,\\ t^{\alpha-1}, & 0\leqslant t\leqslant s.\end{cases}\quad(5)$$

Remark 1 We can easily see that Green's function $G(t,s)$ satisfies

$$G(t,s) \geqslant 0, \ \frac{G(t,s)}{1+t^{\alpha-1}} < \frac{1}{\Gamma(\alpha)}. \tag{6}$$

Proof of Lemma 1: We first show that $\int_0^\infty G(t,s)a(s)y(s)\,\mathrm{d}s \in FC(J,\mathbb{R})$. Since $y(t)$ is bounded on $[0,\infty)$, there exists $M>0$ such that $|y(t)| \leqslant M$, $t \in [0,\infty)$. By (H_3) and (6), we obtain

$$\begin{aligned}\frac{|\int_0^\infty G(t,s)a(s)y(s)\,\mathrm{d}s|}{1+t^{\alpha-1}} &\leqslant \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}a(s)|y(s)|\,\mathrm{d}s\\ &\leqslant \int_0^\infty \frac{M}{\Gamma(\alpha)}a(s)\,\mathrm{d}s\\ &\leqslant \frac{M}{\Gamma(\alpha)}\int_0^\infty a(s)\,\mathrm{d}s < \infty.\end{aligned}$$

Hence $\int_0^\infty G(t,s)a(s)y(s)\,\mathrm{d}s \in FC(J,\mathbb{R})$.

We only transform (3) and (4) as the converse follows by direct computation. From Ref. 5, we know that the general solution of (3) can be written as

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)y(s)\,\mathrm{d}s + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}, \tag{7}$$

where $c_i(i=1,2,\dots n) \in \mathbb{R}$ are arbitrary constants. Because $u(0)=0$, we can see that $c_n=0$. Further, from (7)

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}a(s)y(s)\,\mathrm{d}s + (\alpha-1)c_1t^{\alpha-2} + (\alpha-2)c_2t^{\alpha-3} + \cdots + (\alpha-n+1)c_{n-1}t^{\alpha-n}.$$

Then, from $u'(0)=0$, we obtain $c_{n-1}=0$. Repeating the same steps for $u''(0)=\cdots=u^{n-2}(0)=0$, we obtain $c_{n-2}=c_{n-3}=\cdots=c_2=0$. Considering the condition $D^{\alpha-1}u(\infty)=\beta$, from (7) we obtain

$$D^{\alpha-1}u(\infty) = -\int_0^\infty a(s)y(s)\,\mathrm{d}s + c_1\Gamma(\alpha) = \beta,$$

which yields

$$c_1 = \frac{1}{\Gamma(\alpha)}\left(\int_0^\infty a(s)y(s)\,\mathrm{d}s + \beta\right).$$

Substituting $c_i(i=1,2,\dots n)$ into (7),

$$\begin{aligned}u(t) &= \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty a(s)y(s)\,\mathrm{d}s\\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)y(s)\,\mathrm{d}s\\ &= \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^\infty G(t,s)a(s)y(s)\,\mathrm{d}s,\end{aligned}$$

where $G(t,s)$ is given by (5). □

Next we list some concepts and a fixed point theorem in ordered spaces. Let $(E,\|\cdot\|)$ be a real Banach space, and let $P \subset E$ be a cone. Then E is partially ordered by P, i.e., $x \leqslant y$ if and only if $y-x \in P$. If there is a constant $N>0$ such that for all $x,y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$ then P is called normal. An operator $A: E \to E$ is increasing if $x \leqslant y$ implies $Ax \leqslant Ay$. For x, $y \in E$, the notation $x \sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h>\theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$)) we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. Then $P_h \subset P$.

Lemma 2 (Ref. 21) *Let E be a real Banach space and P be a normal cone in E with $h>\theta$. Operator $A: P \to P$ is increasing and satisfies: (i) there is $h_0 \in P_h$ such that $Ah_0 \in P_h$; (ii) for $x \in P$ and $r \in (0,1)$, there exists $\varphi(r) \in (r,1)$ such that $A(rx) \geqslant \varphi(r)Ax$. Then the operator equation $Ax=x$ has a unique solution x^* in P_h; for $x_0 \in P_h$, constructing the sequence $x_n = Ax_{n-1}$, $n=1,2,\dots$, we obtain $x_n \to x^*$ as $n \to \infty$.*

MAIN RESULTS

We define a cone P in $FC(J,\mathbb{R})$ by

$$P = \{u \in FC(J,\mathbb{R}):\ u(t) \geqslant 0 \text{ on } J\}.$$

Define an integral operator A by

$$Au(t) = \int_0^\infty G(t,s)a(s)f(s,u(s),(Tu)(s))\,\mathrm{d}s + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}, \tag{8}$$

where $G(t,s)$ is given in (5). From Lemma 1, we see that the solution of problem (1) is the fixed point of operator HA.

Theorem 1 *Let $\beta>0$ and (H_1)–(H_4) hold. Then (1) has a unique positive solution u^* in P_h, where $h(t)=t^{\alpha-1}$, $t \in [0,\infty)$. Furthermore, for $u_0 \in P_h$,* *constructing the sequence*

$$u_n(t) = \int_0^\infty G(t,s)a(s)f(s,u_{n-1}(s),(Tu_{n-1})(s))\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}, \quad n = 1,2,\dots,$$

where $G(t,s)$ is given in (5), we have $u_n(t) \to u^*(t)$ as $n \to \infty$.

Proof: First, we prove that the cone P is normal. For u, $v \in P$ with $u \leqslant v$, we have $v - u \in P$. That is, $v(t) - u(t) \geqslant 0$, $t \in J$. Hence $0 \leqslant u(t) \leqslant v(t)$, $t \in J$. Hence $\sup_{t\in J}(u(t)/(1+t^{\alpha-1})) \leqslant \sup_{t\in J} v(t)/(1+t^{\alpha-1})$, Hence $\|u\| \leqslant \|v\|$. So the conclusion holds.

Second, we consider the operator A defined in (8). Now we prove that $A : P \to P$ is increasing. For $u \in P$, we have $u(t) \geqslant 0$, $\|u\| = \sup_{t\in J} u(t)/(1+t^{\alpha-1}) < \infty$. From ($H_2$), we have

$$\begin{aligned}
\sup_{s\in J} \frac{Tu(s)}{1+s^{\alpha-1}} &= \sup_{s\in J} \int_0^s k(s,\tau)\frac{u(\tau)}{1+s^{\alpha-1}}\,d\tau \\
&= \sup_{s\in J} \int_0^s k(s,\tau)\frac{1+\tau^{\alpha-1}}{1+s^{\alpha-1}}\frac{u(\tau)}{1+\tau^{\alpha-1}}\,d\tau \\
&\leqslant \sup_{s\in J} \int_0^s k(s,\tau)\|u\|\,d\tau \\
&= \|u\| \sup_{s\in J} \int_0^s k(s,\tau)\,d\tau \\
&= \|u\| k^* < \infty.
\end{aligned}$$

So there exist constants $L_u, L_T > 0$ such that $\|u\| \leqslant L_u$, $\|Tu\| \leqslant L_T$. From (H_1), there exists $M_1 > 0$ which is defined by

$$M_1 := \sup\{f(s,(1+s^{\alpha-1})u,(1+s^{\alpha-1})v)\}$$

for $s \in [0,\infty)$, $u \in [0,L_u]$, $v \in [0,L_T]$. Also, by (6),

$$\begin{aligned}
\frac{Au(t)}{1+t^{\alpha-1}} &= \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(s,u(s),(Tu)(s))\,ds \\
&\quad + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty a(s)f\bigg(s,(1+s^{\alpha-1}) \\
&\quad \times \frac{u(s)}{1+s^{\alpha-1}},(1+s^{\alpha-1})\frac{(Tu)(s)}{1+s^{\alpha-1}}\bigg)\,ds \\
&\quad + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} \\
&\leqslant \frac{M_1}{\Gamma(\alpha)} \int_0^\infty a(s)\,ds + \frac{\beta}{\Gamma(\alpha)} < \infty.
\end{aligned}$$

From (6), we have $Au(t) \geqslant 0$, $t \in J$, and thus $Au \in P$. Evidently, $T : P \to P$ is increasing. Further, it follows from (H_1) that $A : P \to P$ is increasing.

Take $h(t) = t^{\alpha-1}$, $t \in J$. We show $h \in P$. Indeed, we have $h(t) \geqslant 0$ and

$$\sup_{t\in J} \frac{h(t)}{1+t^{\alpha-1}} = \sup_{t\in J} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} = 1 < \infty.$$

Furthermore, $h \in P_h$. In the following, we prove $Ah \in P_h$. Since

$$\begin{aligned}
\frac{h(s)}{1+s^{\alpha-1}} &= \frac{s^{\alpha-1}}{1+s^{\alpha-1}} \leqslant 1, \\
\frac{Th(s)}{1+s^{\alpha-1}} &\leqslant \int_0^s k(s,\tau)\,d\tau \leqslant k^*.
\end{aligned}$$

Then from (H_1), we can set

$$M_2 = \sup\{f(s,(1+s^{\alpha-1})u,(1+s^{\alpha-1})v)\}$$

for $s \in [0,\infty)$, $u \in [0,1]$, $v \in [0,k^*]$. On the one hand,

$$\begin{aligned}
Ah(t) &= \int_0^\infty G(t,s)a(s)f(s,h(s),(Th)(s))\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} \\
&\geqslant \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}$$

On the other hand, from (H_1) and Remark 1,

$$\begin{aligned}
Ah(t) &= \int_0^\infty G(t,s)a(s)f(s,h(s),(Th)(s))\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} \\
&\leqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty a(s)f\bigg(s,(1+s^{\alpha-1})\frac{h(s)}{1+s^{\alpha-1}}, \\
&\qquad (1+s^{\alpha-1})\frac{Th(s)}{1+s^{\alpha-1}}\bigg)\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} \\
&\leqslant \left(\frac{M_2}{\Gamma(\alpha)} \int_0^\infty a(s)\,ds + \frac{\beta}{\Gamma(\alpha)}\right) t^{\alpha-1}.
\end{aligned}$$

Let

$$\mu_1 = \frac{\beta}{\Gamma(\alpha)},\ \mu_2 = \frac{M_2}{\Gamma(\alpha)} \int_0^\infty a(s)\,ds + \frac{\beta}{\Gamma(\alpha)}.$$

From $\beta > 0$ and (H_3), we have $\mu_2 > \mu_1 > 0$. So $\mu_1 h(t) \leqslant Ah(t) \leqslant \mu_2 h(t)$, $t \in J$. That is, $\mu_1 h \leqslant Ah \leqslant \mu_2 h$ and thus $Ah \in P_h$.

Finally, we prove that the condition (ii) in

Lemma 2 also holds. For $r \in (0,1)$, $u \in P$, from (H_4),

$$\begin{aligned} A(ru)(t) &= \int_0^\infty G(t,s)a(s)f(s,ru(s),(Tru)(s))\,ds \\ &\quad + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} \\ &\geqslant \varphi(r)\int_0^\infty G(t,s)a(s)f(s,u(s),(Tu)(s))\,ds \\ &\quad + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} \\ &\geqslant \varphi(r)\left[\int_0^\infty G(t,s)a(s)f(s,u(s),(Tu)(s))\,ds \right.\\ &\quad \left. + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ &= \varphi(r)Au(t),\ t \in J. \end{aligned}$$

That is, $A(ru) \geqslant \varphi(r)Au$, $r \in (0,1)$, $u \in P$. Hence all the conditions of Lemma 2 are satisfied. Hence the operator equation $Au = u$ has a unique solution u^* in P_h. That is, (1) has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}$, $t \in [0,\infty)$. Furthermore, for $u_0 \in P_h$, constructing a sequence $u_n = Au_{n-1}$, $n = 1,2,\dots$, we obtain $u_n \to u^*$ as $n \to \infty$. That is,

$$u_n(t) = \int_0^\infty G(t,s)a(s)f(s,u_{n-1}(s),(Tu_{n-1})(s))\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}, \quad n = 1,2,\dots,$$

and $u_n(t) \to u^*(t)$ as $n \to \infty$. □

Now we consider problem (2) and give the following conclusion.

Theorem 2 *Let $\beta > 0$. Assume that (H_2), (H_3) hold and*

(H_5) *$f \in C[J \times J \times J \times J, J]$ and $f(t,u,v,w)$ are increasing with respect to $u,v,w \in J$ for fixed $t \in J$; when u,v,w are bounded, $f(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})v,(1+t^{\alpha-1})w)$ is bounded for $t \in J$;*

(H_6) *$h(t,s) \geqslant 0$ and $h^* := \sup_{t\in J}(1/(1+t^{\alpha-1}))\int_0^\infty h(t,s)(1+s^{\alpha-1})\,ds < \infty$;*

(H_7) *for $r \in (0,1)$, there is $\varphi(r) \in (r,1)$ such that $f(t,ru,rv,rw) \geqslant \varphi(r)f(t,u,v,w)$, $\forall t,u,v,w \in J$.*

Then problem (2) has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}$, $t \in [0,\infty)$. Furthermore, for $u_0 \in P_h$, constructing the sequence

$$u_n(t) = \int_0^\infty G(t,s)a(s)f(s,u_{n-1}(s),(Tu_{n-1})(s),(Su_{n-1})(s))\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}, \quad n = 1,2,\dots,$$

where $G(t,s)$ is given in (5), we have $u_n(t) \to u^*(t)$ as $n \to \infty$.

Proof: From the proof of Theorem 1, we only need to prove that $\|Su\| < \infty$ for any $u \in P$. In fact, for $u \in P$, we have $u(t) \geqslant 0$, $\|u\| = \sup_{t\in J} u(t)/(1+t^{\alpha-1}) < \infty$. From ($H_6$), we have

$$\begin{aligned} \|Su\| &= \sup_{s\in J}\frac{Su(s)}{1+s^{\alpha-1}} \\ &= \sup_{s\in J}\frac{1}{1+s^{\alpha-1}}\int_0^\infty h(s,\tau)u(\tau)\,d\tau \\ &= \sup_{s\in J}\frac{1}{1+s^{\alpha-1}}\int_0^\infty h(s,\tau)(1+\tau^{\alpha-1}) \\ &\quad \times \frac{u(\tau)}{1+\tau^{\alpha-1}}\,d\tau \\ &\leqslant \sup_{s\in J}\frac{1}{1+s^{\alpha-1}}\int_0^\infty h(s,\tau)(1+\tau^{\alpha-1})\|u\|\,d\tau \\ &= \|u\|h^* < \infty. \end{aligned}$$

□

Corollary 1 *Let $\beta > 0$ and (H_1), (H_3), (H_4), (H_6) hold. Then*

$$\left.\begin{aligned} &D^\alpha u(t) + a(t)f(t,u(t),Su(t)) = 0, \\ &n-1 < \alpha \leqslant n,\ n \in \mathbb{N},\ n \geqslant 2, \\ &u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\ &D^{\alpha-1}u(\infty) = \beta, \end{aligned}\right\} \tag{9}$$

where $t \in J$ and S is as in (2), has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}$, $t \in [0,\infty)$. Furthermore, for $u_0 \in P_h$, constructing the sequence

$$u_n(t) = \int_0^\infty G(t,s)a(s)f(s,u_{n-1}(s),(Su_{n-1})(s))\,ds + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}, \quad n = 1,2,\dots,$$

where $G(t,s)$ is given in (5), we have $u_n(t) \to u^*(t)$ as $n \to \infty$.

Corollary 2 *Let $\beta > 0$. Assume that (H_3) holds and*

(H_8) *$f \in C[J \times J, J]$ and $f(t,u)$ is increasing with respect to the second variable and when u is bounded, $f(t,(1+t^{\alpha-1})u)$ is bounded for $t \in J$;*

(H_9) *for $r \in (0,1)$, there exists $\varphi(r) \in (r,1)$ such that $f(t,ru) \geqslant \varphi(r) f(t,u)$, $\forall t,\ u \in J$.*

Then

$$\left.\begin{aligned} D^{\alpha}u(t)+a(t)f(t,u(t))=0,\\ n-1<\alpha\leqslant n,\ n\in\mathbb{N},\ n\geqslant 2,\\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0,\\ D^{\alpha-1}u(\infty)=\beta, \end{aligned}\right\} \tag{10}$$

where $t \in J$, has a unique positive solution u^* in P_h, where $h(t)=t^{\alpha-1}$, $t \in [0,\infty)$. Furthermore, for $u_0 \in P_h$, constructing the sequence

$$u_n(t)=\int_0^{\infty} G(t,s)a(s)f(s,u_{n-1}(s))\,\mathrm{d}s + \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)}, \quad n=1,2,\ldots,$$

where $G(t,s)$ is given in (5), we have $u_n(t) \to u^*(t)$ as $n \to \infty$.

AN EXAMPLE

Consider the problem

$$\left.\begin{aligned} D^{5/2}u(t)+\mathrm{e}^{-t}\left\{\sin\frac{\pi t}{1+t}+\frac{u^{1/3}(t)}{1+t^{3/2}}\right.\\ \left.+\frac{1}{1+t^{3/2}}\left(\int_0^t\frac{u(s)}{(1+t+s)^2}\,\mathrm{d}s\right)^{1/4}\right\}=0,\\ u(0)=u'(0)=0,\quad D^{3/2}u(\infty)=1. \end{aligned}\right\} \tag{11}$$

The problem (11) can be regarded as a form of (1), where $\alpha=\frac{5}{2}$, $a(t)=\mathrm{e}^{-t}$, $k(t,s)=1/(1+t+s)^2$, $n=3$, $\beta=1$ and

$$f(t,u,v)=\sin\frac{\pi t}{1+t}+\frac{u^{1/3}}{1+t^{3/2}}+\frac{v^{1/4}}{1+t^{3/2}},$$

$$Tu(t)=\int_0^t\frac{u(s)}{(1+t+s)^2}\,\mathrm{d}s.$$

It is clear that $f \in C(J\times J\times J, J)$, and f increases with respect to the second and third variables. If $u,v \in [0,\infty)$ are bounded, then

$$\begin{aligned} &f(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})v)\\ &\quad=f(t,(1+t^{3/2})u,(1+t^{3/2})v)\\ &\quad=\sin\frac{\pi t}{1+t}+u^{1/3}+v^{1/4}\\ &\quad\leqslant 1+u^{1/3}+v^{1/4}, \end{aligned}$$

which implies that $f(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})v)$ is bounded. In addition,

$$\begin{aligned} k^* &= \sup_{t\geqslant 0}\int_0^t\frac{1}{(1+t+s)^2}\,\mathrm{d}s=\sup_{t\geqslant 0}\frac{t}{(1+t)(1+2t)}\\ &=3-2\sqrt{2}\approx 0.17, \end{aligned}$$

$$\int_0^{\infty} a(s)\,\mathrm{d}s=\int_0^{\infty}\mathrm{e}^{-s}\,\mathrm{d}s=1.$$

Let $\varphi(r)=r^{1/3}$, $r\in(0,1)$. Then $\varphi(r)\in(r,1)$ and, for $t\in J$, $u,\ v\in J$,

$$\begin{aligned} f(t,ru,rv)&=\sin\frac{\pi t}{1+t}+\frac{r^{1/3}u^{1/3}}{1+t^{3/2}}+\frac{r^{1/4}v^{1/4}}{1+t^{3/2}}\\ &\geqslant r^{1/3}\left(\sin\frac{\pi t}{1+t}+\frac{u^{1/3}}{1+t^{3/2}}+\frac{v^{1/4}}{1+t^{3/2}}\right)\\ &=\varphi(r)f(t,u,v). \end{aligned}$$

Hence all the conditions of Theorem 1 are satisfied. By Theorem 1, problem (11) has a unique positive solution u^* in P_h, where $h(t)=t^{\alpha-1}=t^{3/2}$, $t\in[0,\infty)$. In addition, for the sequence

$$\begin{aligned} u_n(t)=\int_0^{\infty}G(t,s)\,\mathrm{e}^{-s}\Bigg\{\sin\frac{\pi s}{1+s}+\frac{[u_{n-1}(s)]^{1/3}}{1+s^{3/2}}\\ +\frac{1}{1+s^{3/2}}\left(\int_0^s\frac{u_{n-1}(\tau)}{(1+s+\tau)^2}\,\mathrm{d}\tau\right)^{1/4}\Bigg\}\,\mathrm{d}s+\frac{t^{3/2}}{\Gamma(\alpha)}, \end{aligned}$$

$n=1,2,\ldots$, for $u_0\in P_{t^{3/2}}$, we have $u_n(t)\to u^*(t)$ as $n\to\infty$, where

$$\Gamma(\tfrac{5}{2})G(t,s)=\begin{cases} t^{3/2}-(t-s)^{3/2}, & 0\leqslant s\leqslant t,\\ t^{3/2}, & 0\leqslant t\leqslant s. \end{cases}$$

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