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Stability of Fréchet's functional equation on certain groupoids

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ABSTRACT: We obtain a Hyers-Ulam stability result for Fréchet's functional equation $\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} f(x) = 0$ for all $x, y_1, y_2, \dots, y_{n+1} \in G$, where $f : G \to B$ is a mapping from a power-associative, mth-power-symmetric groupoid with a left identity into a real Banach space, and m, n are nonnegative integers with $m > 1$.

KEYWORDS: bounded difference, Hyers-Ulam stability

MSC2010: 39B82

INTRODUCTION

The stability problems for functional equations arose when Ulam proposed the following question[1]. Suppose that a function f satisfies the Cauchy (additive) functional equation

$$f(x+y) = f(x) + f(y)$$

only approximately. Then does there exist an additive function which approximates f? This question has been partially answered by Hyers[2]. Hyers' statement is as follows. Let E and E' be Banach spaces, δ be a positive number and $f : E \to E'$ satisfying

$$\|f(x+y) - f(x) - f(y)\| < \delta$$

for all $x, y \in E$. Then the limit $l(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for each $x \in E$. Furthermore, l is a unique additive function satisfying $\|f(x) - l(x)\| \leqslant \delta$ for all $x \in E$. Thus any result on functions which satisfies a functional equation 'only approximately' (with bounded errors) will be referred to as a Hyers-Ulam stability result of the respective functional equation. When such a function can always be approximated with an actual solution of the functional equation, we say that the functional equation is stable[3]. Stability of functional equations related to Cauchy functional equation have been widely studied. The *monomial functional equation*

$$\Delta^n_{y,y,\dots,y} f(x) - n! f(y) = 0 \tag{1}$$

for all $x, y \in E$, and *Fréchet's functional equation*

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} f(x) = 0 \tag{2}$$

for all $x, y_1, y_2, \dots, y_{n+1} \in E$ and their stability problems have been studied by various authors[4–6]. One of the most general Hyers-Ulam stability results for (1) is the work of Gilányi[7]. His statement is as follows. Let $n \geqslant 1$ and $m \geqslant 2$ be integers, G be a power-associative, mth-power-symmetric groupoid, B be a Banach space, and $f : G \to B$ be a function. If there exists a nonnegative real number ϵ for which

$$\|\Delta^n_{y,y,\dots,y} f(x) - n! f(y)\| \leqslant \epsilon$$

for all $x, y \in G$, then there exists a unique function $g : S \to B$ such that $\Delta^n_{y,y,\dots,y} g(x) - n! g(y) = 0$ for all $x, y \in G$ and

$$\|f(x) - g(x)\| \leqslant \frac{1}{n!}\epsilon$$

for all $x \in G$.

It is not hard to see that (1) implies $\Delta^{n+1}_{y,y,\dots,y} f(x) = 0$ for all $x, y \in E$, which is equivalent to (2) in many cases[4,8,9]. Hence we are inspired to study the stability of Fréchet's functional equation for the case when the domain of f is a power-associative, power-symmetric groupoid.

BACKGROUND

Difference operator

Let $\mathbb{N}$ be the set of positive integers, B be a real Banach space, and $(G, \circ)$ be a groupoid. We use the notation

$$x_1 \circ x_2 \circ \cdots \circ x_n = (\cdots((x_1 \circ x_2) \circ x_3) \circ \cdots) \circ x_n,$$

that is, $\circ$ is a left-to-right operation. The powers of $x \in G$ are then defined by

$$x^k = \underbrace{x \circ x \circ \cdots \circ x}_{k \text{ terms}}.$$

For each $f : G \to B$, the difference operator Δ and its iterates are defined by $\Delta_y f(x) = f(x \circ y) - f(x)$ and

$$\Delta^{k+1}_{y_1,y_2,\dots,y_{k+1}} f(x) = \Delta_{y_1}(\Delta^k_{y_2,\dots,y_k} f)(x)$$

for all $k \in \mathbb{N}$ and $x, y, y_1, y_2, \dots, y_k \in G$, and $\Delta^1 = \Delta$. It can be shown that

$$\Delta^k_{y_1,y_2,\dots,y_k} f(x) = \sum_{i_1,i_2,\dots,i_k \in \{0,1\}} \Big[(-1)^{k-i_1-i_2-\dots-i_k} \cdot f(x \circ y_1^{i_1} \circ y_2^{i_2} \circ \cdots \circ y_k^{i_k})\Big]$$

when $x \circ y^0 := x$ for all $x, y \in G$.

For brevity, we denote $s_k = k - i_1 - i_2 - \cdots - i_k$, and

$$\mathscr{S}_k F(i_1, i_2, \dots, i_k) := \sum_{i_1,i_2,\dots,i_k \in \{0,1\}} (-1)^{s_k} F(i_1, i_2, \dots, i_k)$$

for each $k \in \mathbb{N}$. Hence

$$\Delta^k_{y_1,y_2,\dots,y_k} f(x) = \mathscr{S}_k f(x \circ y_1^{i_1} \circ y_2^{i_2} \circ \cdots \circ y_k^{i_k}). \quad (3)$$

Also note that

$$\Delta^k_{y_1,y_2,\dots,y_k}(f_1 + f_2)(x) = \Delta^k_{y_1,y_2,\dots,y_k} f_1(x) + \Delta^k_{y_1,y_2,\dots,y_k} f_2(x) \quad (4)$$

for any $f_1, f_2 : G \to B$, where

$$(f_1 + f_2)(x) := f_1(x) + f_2(x).$$

From now on,

$$\Delta^k_{y_1,y_2,\dots,y_k}(f_1(x) + f_2(x)) := \Delta^k_{y_1,y_2,\dots,y_k}(f_1 + f_2)(x).$$

Fréchet's functional equation on commutative semigroups

To give the solutions of (2) when the domain of f is a commutative semigroup S, we introduce the notion of additivity of functions on S. A function $A : S \to B$ is *additive* if $A(xy) = A(x) + A(y)$ for all $x, y \in S$. Note that if A is additive, then $A(x^k) = kA(x)$ for all $k \in \mathbb{N}$ and $x \in S$.

For $n \in \mathbb{N}$, an *n-additive function* is a function $A_n : S^n \to B$ that is additive with respect to each argument (when the other arguments are fixed). The *diagonalization* of A_n is a function $A^n : S \to B$ such that $A^n(x) = A_n(x, x, \dots, x)$ for all $x \in S$. It is not hard to see that

$$A^n(x^k) = k^n A^n(x) \quad (5)$$

for all $k \in \mathbb{N}$ and $x \in S$. Additionally, $A^n(e) = 0$ when S has an identity element e.

Any solution of (2) (when the domain of f is a semigroup) is called a *generalized polynomial*. They are described by the following theorem.

Theorem 1 (Theorem C of Ref. 4) *Let S be a commutative semigroup. Assume that $f : S \to B$ and $n \in \mathbb{N}$ satisfy $\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} f(x) = 0$ for all $x, y_1, y_2, \dots, y_{n+1} \in S$. Then there exist $c \in B$ and $A^1, A^2, \dots, A^n : S \to B$ such that each A^k is the diagonalization of a k-additive function and*

$$f(x) = c + A^1(x) + A^2(x) + \cdots + A^n(x)$$

for every $x \in S$.

The next theorem gives a stability result for Fréchet's functional equation on commutative semigroups.

Theorem 2 (Theorem 3 of Ref. 4) *Let S be a commutative semigroup. Assume that $f : S \to B$, $n \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$ satisfy the inequality $\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} f(x)\| \leqslant \epsilon$ for all $x, y_1, y_2, \dots, y_{n+1} \in S$. Then there exists a generalized polynomial $P : S \to B$ given by*

$$P(x) = \sum_{k=1}^{n} A^k(x),$$

where each A^k is the diagonalization of a k-additive function on S, such that $\|\Delta_y(f(x) - P(x))\| \leqslant \epsilon$ for all $x, y \in G$.

Power associativity and power symmetry

Following the definitions used by Gilányi[7], the groupoid $(G, \circ)$ is *power associative* if $x^k \circ x^l = x^{k+l}$ for every $x \in G$ and for all $k, l \in \mathbb{N}$. It is not hard to see that, if $(G, \circ)$ is power associative, then $x^{kl} = (x^k)^l$ for all $x \in G$ and for all $k, l \in \mathbb{N}$. For an integer $m > 1$, $(G, \circ)$ is *mth power symmetric* if $(x \circ y)^m = x^m \circ y^m$ for all $x, y \in G$. It can be shown via induction that if $(G, \circ)$ is power associative and mth power symmetric, it is also m^sth power symmetric for all $s \in \mathbb{N}$.

RESULTS

The following example shows that power associativity and power symmetry of the domain are not sufficient to imply stability of Fréchet's functional equation.

Example 1 Consider $(\mathbb{N},\circ)$ where

$$(m\circ k)=\begin{cases} m, & m\leqslant k,\\ m-1, & m>k.\end{cases}$$

Then $(\mathbb{N},\circ)$ is a power associative (since $m^k=m$ for all $m,k\in\mathbb{N}$), kth-power symmetric for all $k>1$ (since $(m_1\circ m_2)^k=m_1\circ m_2=m_1^k\circ m_2^k$ for all $m_1,m_2,k\in\mathbb{N}$). Define $f:\mathbb{N}\to\mathbb{R}$ by $f(k)=k/(n+1)2^{n+1}$ for all $k\in\mathbb{N}$. Then by (3),

$$\begin{aligned}
&(n+1)2^{n+1}\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}f(x)\|\\
&\quad=(n+1)2^{n+1}\|\mathscr{S}_{n+1}f(x\circ y_1^{i_1}\circ y_2^{i_2}\circ\cdots\circ y_{n+1}^{i_{n+1}})\|\\
&\quad=\|\mathscr{S}_{n+1}(x\circ y_1^{i_1}\circ y_2^{i_2}\circ\cdots\circ y_{n+1}^{i_{n+1}})\|\\
&\quad=\|\mathscr{S}_{n+1}x-\mathscr{S}_{n+1}(x-x\circ y_1^{i_1}\circ y_2^{i_2}\circ\cdots\circ y_{n+1}^{i_{n+1}})\|\\
&\quad\leqslant\sum_{i_1,_2,\dots,i_{n+1}\in\{0,1\}}\|x-x\circ y_1^{i_1}\circ y_2^{i_2}\circ\cdots\circ y_{n+1}^{i_{n+1}}\|\\
&\quad\leqslant\sum_{i_1,_2,\dots,i_{n+1}\in\{0,1\}}(n+1)=(n+1)2^{n+1}
\end{aligned}$$

since $x-x\circ y_1^{i_1}\circ y_2^{i_2}\circ\cdots\circ y_{n+1}^{i_{n+1}}\in\{0,1,2,\dots,n+1\}$ for each set of $i_1,i_2,\dots,i_{n+1}$. Hence

$$\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}f(x)\|\leqslant 1.$$

This holds for all $x,y_1,y_2,\dots,y_{n+1}\in\mathbb{N}$. But if a function $g:\mathbb{N}\to\mathbb{R}$ satisfies $\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}g(x)=0$ (with respect to operation $\circ$) for all $x,y_1,y_2,\dots,y_{n+1}\in\mathbb{N}$, then

$$\begin{aligned}
&\Delta^{n+1}_{m,m,\dots,m}g(m+1)\\
&\quad=\mathscr{S}_{n+1}g((m+1)\circ m^{i_1}\circ\cdots\circ m^{i_{n+1}})\\
&\quad=\sum_{i_1,i_2,\dots,i_{n+1}\in\{0,1\}}\big[(-1)^{s_{n+1}}\\
&\qquad\qquad\cdot g((m+1)\circ m^{i_1}\circ\cdots\circ m^{i_{n+1}})\big]\\
&\quad=(-1)^{n+1}g(m+1)\\
&\qquad+\sum_{(i_1,i_2,\dots,i_{n+1})\in I^*}(-1)^{n+1-i_1-i_2-\dots-i_{n+1}}g(m)\\
&\quad=(-1)^{n+1}g(m+1)-(-1)^{n+1}g(m)\\
&\quad=(-1)^{n+1}(g(m+1)-g(m))
\end{aligned}$$

for all $m\in\mathbb{N}$, where

$$I^*=\{0,1\}^{n+1}\setminus\{(\underbrace{0,0,\dots,0}_{n+1\text{ terms}})\}.$$

Since $\Delta^{n+1}_{m,m,\dots,m}g(m+1)=0$, we have $g(m+1)-g(m)=0$ for all $m\in\mathbb{N}$, that is, g is a constant map. Hence $\|f(x)-g(x)\|$ is not bounded. This implies that Fréchet's functional equation is not stable on $(\mathbb{N},\circ)$ with any order n.

Fréchet's functional equation on the set of nonnegative integers

From Theorem 1 and Theorem 2 the following corollaries are obtained.

Corollary 1 *Assume that $f:\mathbb{N}\cup\{0\}\to B$ and $n\in\mathbb{N}$ satisfy*

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}f(x)=0$$

(with respect to addition) for all $x,y_1,y_2,\dots,y_{n+1}\in\mathbb{N}\cup\{0\}$. Then there exist $c,a_1,a_2,\dots,a_n\in B$ such that

$$f(x)=c+xa_1+x^2a_2+\cdots+x^na_n$$

for every $x\in\mathbb{N}\cup\{0\}$.

Proof: By Theorem 1, there exists $P:\mathbb{N}\cup\{0\}\to B$ given by

$$f(x)=c+A^1(x)+A^2(x)+\cdots+A^n(x)$$

where each A^k is the diagonalization of a k-additive function. It follows by (5) that

$$f(x)=c+xA^1(1)+x^2A^2(1)+\cdots+x^nA^n(1)$$

for every $x\in\mathbb{N}\cup\{0\}$. □

Corollary 2 *Let $n\in\mathbb{N}$. Assume that $f:\mathbb{N}\cup\{0\}\to B$ and $\epsilon\in\mathbb{R}^+$ satisfy the inequality*

$$\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}f(x)\|\leqslant\epsilon$$

with respect to addition for all $x,y_1,y_2,\dots,y_{n+1}\in\mathbb{N}\cup\{0\}$. Then there exists a generalized polynomial $P:\mathbb{N}\cup\{0\}\to B$ such that

$$P(0)=0$$

and

$$\|f(x)-f(0)-P(x)\|\leqslant\epsilon$$

for all $x\in\mathbb{N}\cup\{0\}$. Moreover, P is given by

$$P(x)=\sum_{i=1}^{n}x^ia_i,$$

for all $x\in\mathbb{N}\cup\{0\}$, where $a_1,a_2,\dots,a_n\in B$.

Proof: The result follows from Theorem 2 with a similar proof to that of Corollary 1. □

Remark 1 From Corollary 2, we denote $E(x) = f(x) - P(x) - f(0)$. Then E is a bounded function and $f(x) = f(0) + E(x) + P(x)$. Since E is bounded, $\lim_{k\to\infty}(f(0) + E(kx))/k = 0$ for all $x \in \mathbb{N} \cup \{0\}$. Note that

$$P(x) = \sum_{i=1}^{n} x^i a_i.$$

Hence $a_1, a_2, \dots, a_n$ can be found recursively by

$$\lim_{k\to\infty} \frac{1}{k^n} f(k) = \lim_{k\to\infty} \left(\frac{f(0)+E(k)}{k^n} + \frac{\sum_{i=1}^{n} k^i a_i}{k^n} \right) = a_n$$

and

$$\lim_{k\to\infty} \frac{f(k) - \sum_{i=m+1}^{n} k^i a_1}{k^m} = \lim_{k\to\infty} \left(\frac{f(0)+E(k)}{k^m} + \frac{\sum_{i=1}^{m} k^i a_i}{k^m} \right) = a_m$$

for each $m \in \{1, 2, \dots, n-1\}$.

Stability of Fréchet's functional equation on groupoids

The main result shows that, with the existence of a left identity element, we can reduce associativity and commutativity to power associativity and power symmetry, respectively, without affecting the stability of Fréchet's functional equation. The following lemma will be used repeatedly in the main theorem.

Lemma 1 *Let G be a power-associative, mth-power-symmetric (with $m > 1$) groupoid, B be a Banach space, and n be a nonnegative integer. Assume that $\epsilon \in \mathbb{R}^+$ and $f : G \to B$ satisfy*

$$\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} f(x)\| \leqslant \epsilon$$

for all $x, y_1, y_2, \dots, y_{n+1} \in G$. If k is a positive integer such that the limit

$$L(x) := \lim_{s\to\infty} \frac{1}{m^{sk}} f(x^{m^s})$$

exists for every $x \in G$, then

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} L(x) = 0$$

for all $x, y_1, y_2, \dots, y_{n+1} \in G$.

Proof: Assume that all the assumptions are met. Observe that

$$\begin{aligned}
\left\| \frac{1}{m^{sk}} \mathscr{S}_{n+1} f(x^{m^s} \circ y_1^{m^s i_1} \circ y_2^{m^s i_2} \circ \dots \circ y_{n+1}^{m^s i_{n+1}}) \right\| &= \left\| \frac{1}{m^{sk}} \Delta^{n+1}_{y_1^{m^s}, y_2^{m^s}, \dots, y_{n+1}^{m^s}} f(x^{m^s}) \right\| \\
&\leqslant \frac{1}{m^{sk}} \epsilon.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} L(x)\| \\
&= \|\mathscr{S}_{n+1} L(x \circ y_1^{i_1} \circ y_2^{i_2} \circ \dots \circ y_{n+1}^{i_{n+1}})\| \\
&= \left\| \mathscr{S}_{n+1} \lim_{s\to\infty} \frac{1}{m^{sk}} f((x \circ y_1^{i_1} \circ y_2^{i_2} \circ \dots \circ y_{n+1}^{i_{n+1}})^{m^s}) \right\| \\
&= \left\| \lim_{s\to\infty} \frac{1}{m^{sk}} \mathscr{S}_{n+1} f((x \circ y_1^{i_1} \circ y_2^{i_2} \circ \dots \circ y_{n+1}^{i_{n+1}})^{m^s}) \right\| \\
&\leqslant \lim_{s\to\infty} \frac{1}{m^{sk}} \epsilon \\
&= 0.
\end{aligned}$$

This follows from

$$f((x \circ y_1^{i_1} \circ y_2^{i_2} \circ \dots \circ y_{n+1}^{i_{n+1}})^{m^s}) = f(x^{m^s} \circ y_1^{m^s i_1} \circ y_2^{m^s i_2} \circ \dots \circ y_{n+1}^{m^s i_{n+1}}).$$

□

Now we are ready to establish the main result.

Theorem 3 *Let G be a power-associative, mth-power-symmetric (with $m > 1$) groupoid with a left identity e, B be a Banach space, and n be a nonnegative integer. Assume that $\epsilon \in \mathbb{R}^+$ and $f : G \to B$ satisfy $\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} f(x)\| \leqslant \epsilon$ for all $x, y_1, y_2, \dots, y_{n+1} \in G$. Then there exists a unique function $P : G \to B$ such that P satisfies*

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}} P(x) = 0 \tag{6}$$

for all $x, y_1, y_2, \dots, y_{n+1} \in G$, $P(e) = 0$ and $\|f(x) - f(e) - P(x)\| \leqslant \epsilon$ for all $x \in G$.

Proof: The case where $n = 0$ is obvious, so assume that $n > 0$. For each $x \in G$, define $F_x : \mathbb{N} \cup \{0\} \to B$ by $F_x(0) = f(e)$ and $F_x(s) = f(x^s)$ for each $s \in \mathbb{N}$. Since G is power associative,

$$\begin{aligned}
&\|\Delta^{n+1}_{t_1,t_2,\dots,t_{n+1}} F_x(s)\| \\
&= \|\mathscr{S}_{n+1} F_x(s + i_1 t_1 + i_2 t_2 + \dots + i_{n+1} t_{n+1})\| \\
&= \|\mathscr{S}_{n+1} f(x^{s + i_1 t_1 + i_2 t_2 + \dots + i_{n+1} t_{n+1}})\| \\
&= \|\mathscr{S}_{n+1} f(x^s \circ x^{i_1 t_1} \circ \dots \circ x^{i_{n+1} t_{n+1}})\| \\
&= \|\Delta^{n+1}_{x^{t_1}, x^{t_2}, \dots, x^{t_{n+1}}} f(x^s)\| \leqslant \epsilon
\end{aligned}$$

and

$$\begin{aligned}&\|\Delta^{n+1}_{t_1,t_2,\dots,t_{n+1}}F_x(0)\|\\&=\|\mathscr{S}_{n+1}F_x(i_1t_1+i_2t_2+\cdots+i_{n+1}t_{n+1})\|\\&=\|\mathscr{S}_{n+1}f(e\circ x^{i_1t_1+i_2t_2+\cdots+i_{n+1}t_{n+1}})\|\\&=\|\mathscr{S}_{n+1}f(e\circ x^{i_1t_1}\circ x^{i_2t_2}\circ\cdots\circ x^{i_{n+1}t_{n+1}})\|\\&=\|\Delta^{n+1}_{x^{t_1},x^{t_2},\dots,x^{t_{n+1}}}f(e)\|\leqslant\epsilon\end{aligned}$$

for all $s,t_1,t_2,\dots,t_{n+1}\in\mathbb{N}$. By Corollary 2 and Remark 1, $F_x(s)=E_x(s)+F_x(0)+sa_{x,1}+s^2a_{x,2}+\cdots+s^na_{x,n}$, where

$$\left.\begin{aligned}a_{x,n}&=\lim_{k\to\infty}\frac{1}{k^n}F_x(k)=\lim_{s\to\infty}\frac{1}{m^{sn}}F_x(m^s),\\a_{x,n-1}&=\lim_{k\to\infty}\frac{1}{k^{n-1}}(F_x(k)-k^na_{x,n})\\&=\lim_{s\to\infty}\frac{1}{m^{s(n-1)}}(F_x(m^s)-(m^{sn})a_{x,n}),\\&\vdots\\a_{x,1}&=\lim_{k\to\infty}\frac{1}{k}\left(F_x(k)-\sum_{i=2}^{n}k^ia_{x,i}\right)\\&=\lim_{s\to\infty}\frac{1}{m^s}\left(F_x(m^s)-\sum_{i=2}^{n}m^{si}a_{x,i}\right),\end{aligned}\right\}\quad(7)$$

and $\|E_x(s)\|\leqslant\epsilon$ for all $s\in\mathbb{N}\cup\{0\}$. Note that

$$\begin{aligned}&a_{x^t,n}\\&=\lim_{s\to\infty}\frac{1}{m^{sn}}F_{x^t}(m^s)\\&=\lim_{s\to\infty}\frac{1}{m^{sn}}f((x^t)^{m^s})\\&=\lim_{s\to\infty}\frac{1}{m^{sn}}f((x^{tm^s}))\\&=\lim_{s\to\infty}\frac{1}{m^{sn}}F_x(tm^s)\\&=\lim_{s\to\infty}\frac{1}{m^{sn}}\left(E(tm^s)+F_x(0)+\sum_{i=1}^{n}(tm^s)^ia_{x,i}\right)\\&=t^na_{x,n}.\end{aligned}$$

It can be shown in similar way that

$$a_{x^t,k}=t^ka_{x,k}\tag{8}$$

for all $x\in G$ for all $t\in\mathbb{N}$ and for all $k\in\{1,2,\dots,n\}$. For each $k\in\{1,2,\dots,n\}$, let $A^k(x)=a_{x,k}$. Define $P:G\to B$ by $P(x)=\sum_{i=1}^{n}A^i(x)$ for all $x\in G$. Next we will show that P satisfies (6) for all $x,y_1,y_2,\dots,y_{n+1}\in G$.

From the definition of A^n and (7),

$$A^n(x)=\lim_{s\to\infty}\frac{1}{m^{sn}}f(x^{m^s}).\tag{9}$$

By the definition of A^k and using (7) and (8), it can be shown that

$$A^k(x)=\lim_{s\to\infty}\frac{1}{m^{sk}}\left(f(x^{m^s})-\sum_{i=k+1}^{n}A^i(x^{m^s})\right)\tag{10}$$

for all $k\in\{1,2,\dots,n-1\}$. From Lemma 1 and (9), $\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}A^n(x)=0$ for all $x,y_1,y_2,\dots,y_{n+1}\in G$. From (4),

$$\begin{aligned}&\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}(f(x)-A^n(x))\|\\&\qquad=\|\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}f(x)-\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}A^n(x)\|\leqslant\epsilon\end{aligned}$$

for all $x,y_1,y_2,\dots,y_{n+1}\in G$. By (10), applying Lemma 1 recursively results in

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}A^k(x)=0$$

for all $x,y_1,y_2,\dots,y_{n+1}\in G$ and for every $k\in\{1,2,\dots,n-1\}$. Hence

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}P(x)=\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}\sum_{i=1}^{n}A^i(x)=0$$

for all $x,y_1,y_2,\dots,y_{n+1}\in G$.

Also, for each $x\in G$

$$\begin{aligned}&\|f(x)-f(e)-P(x)\|\\&=\left\|\left(f(x)-\sum_{i=1}^{n}A^i(x)\right)-f(e)\right\|\\&=\left\|\left(F_x(1)-\sum_{i=1}^{n}a_{x,i}\right)-f(e)\right\|\\&=\|(E_x(1)+F_x(0))-f(e)\|\\&=\|E_x(1)+f(e)-f(e)\|\\&=\|E_x(1)\|\leqslant\epsilon.\end{aligned}$$

By the definitions of A^k and F_e, we have $A^k(e)=0$ for every $k\in\{1,2,\dots,n\}$. So $P(e)=0$.

For the uniqueness of P, let $P^*:G\to B$ satisfy (6), $P^*(e)=0$ and $\|f(x)-f(e)-P^*(x)\|\leqslant\epsilon$ for all $x,y\in G$. Then we have

$$\|P(x)-P^*(x)\|\leqslant 2\epsilon$$

for all $x\in G$. We also have

$$\Delta^{n+1}_{y_1,y_2,\dots,y_{n+1}}(P(x)-P^*(x))=0$$

for all $x,y_1,y_2,\dots,y_{n+1}\in G$, since both P and P^* satisfy (6).

Now let $x \in G$. We will show that $P(x)–P^*(x) = 0$. Define $H : \mathbb{N} \cup \{0\} \to B$ by

$$H(0) = P(e) - P^*(e) = 0 - 0 = 0,$$
$$H(s) = P(x^s) - P^*(x^s)$$

for every $s \in \mathbb{N}$. Since $P - P^*$ satisfies (6), it can be shown by an argument similar to the beginning of the proof that

$$\Delta^{n+1}_{t_1,t_2,\dots,t_{n+1}} H(s) = 0$$

for all $s, t_1, t_2, \dots, t_{n+1} \in \mathbb{N} \cup \{0\}$. By Corollary 1

$$H(s) = c + s b_1 + s^2 b_2 + \dots + s^n b_n$$

for all $s \in \mathbb{N} \cup \{0\}$, where $c, b_1, b_2, \dots b_n \in B$.

Since $\|P(x^s) - P^*(x^s)\| \leqslant 2\epsilon$ for all $s \in \mathbb{N}$ and $P(e) - P^*(e) = 0$, we have $\|H(s)\| \leqslant 2\epsilon$ for all $s \in \mathbb{N} \cup \{0\}$. Hence $\|\Delta^i_t H(s)\| \leqslant 2^{i+1}\epsilon$ for all $s, t \in \mathbb{N} \cup \{0\}$ and for all $i \in \{1, 2, \dots, n\}$. Suppose that H is not a constant function and there exists the largest $k \in \{1, 2, \dots, n\}$ such that $b_k \neq 0$. It is straightforward to show that $\Delta^k_t H(0) = k!t^k b_k$ for all $t \in \mathbb{N} \cup \{0\}$. So $\|k!t^k b_i\| \leqslant 2^{i+1}\epsilon$, that is,

$$\|b_k\| \leqslant \frac{2^{k+1}\epsilon}{k!t^k}$$

for all $t \in \mathbb{N}$. This implies $\|b_k\| = 0$, a contradiction. Hence H is a constant function. So

$$P(x) - P^*(x) = H(1) = H(0) = P(e) - P^*(e) = 0.$$

□

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