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Superconvergence of triangular mixed finite element methods for nonlinear optimal control problems

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ABSTRACT: In this paper, we investigate the superconvergence of nonlinear elliptic optimal control problems by using triangular mixed finite element methods. The state and the co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We obtain the superconvergence of $O(h^{3/2})$ for the control variable and coupled state variable. Numerical results demonstrating these superconvergence results are also presented.

KEYWORDS: lowest order Raviart-Thomas mixed finite element methods, triangular partition

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INTRODUCTION

Optimal control problems play increasingly important role in multi-disciplinary applications such as engineering design, fluid mechanics, physical, biological, medicine, finance, and social-economic systems. There are various numerical methods to solve these complex problems. Among these numerical methods, finite element methods for state equations have many applications. Papers devoted to linear-quadratic optimal control problems include those by Falk[1] and Geveci[2]. The authors studied the numerical approximation of distributed nonlinear optimal control problems with pointwise constraints on the control[3]. Meyer and Rösch[4] analysed finite element discretization of the dimensional (2-d) elliptic optimal control problem. These approximations have convergence of order h^2. A posteriori error estimates for distributed convex optimal control problems and nonlinear optimal control problems have been obtained[5,6].

Compared with standard finite element methods, mixed finite element methods have many advantages. In many control problems, the objective functional contains the gradient of the state variables. Thus the accuracy of the gradient is important in the numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods.

Recently, we obtained a priori error estimates and a posteriori error estimates of mixed finite element methods for linear and nonlinear optimal control problems[7–9]. Then we used the post-processing projection operator to prove a quadratic superconvergence of the control for linear elliptic optimal control problem by a mixed finite element method[10–12].

We are concerned with the 2-d nonlinear elliptic optimal control problem

$$\min_{u\in U_{\mathrm{ad}}}\left\{\tfrac{1}{2}\,\|\mathbf{p}-\mathbf{p}_{\mathrm{d}}\|^2+\tfrac{1}{2}\,\|y-y_{\mathrm{d}}\|^2+\frac{\alpha}{2}\,\|u\|^2\right\} \quad (1)$$

subject to the state equations

$$\operatorname{div}\mathbf{p}+\phi(y)=u,\ \mathbf{p}=-A\nabla y, \quad x\in\Omega, \quad (2)$$

with the boundary condition

$$y=0, \quad x\in\partial\Omega, \quad (3)$$

where Ω is a rectangular domain, $\mathbf{p}_{\mathrm{d}}$ and y_{d} are two known functions, $\mathbf{p}$ and y are state variables, u is a control variable, and $\nu > 0$ is a constant. We denote the set of admissible controls by U_{ad}, where

$$U_{\mathrm{ad}} = \{u \in L^2(\Omega) : u \geqslant 0 \text{ a.e. in } \Omega\}.$$

Let us state the assumptions on the operator A and the functional ϕ: (A1) the coefficient matrix function $A(x) = (a_{ij}(x))$ is symmetric with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition $c_*|\xi|^2 \leqslant \sum_{i,j=1}^2 a_{ij}(x)\xi_i\xi_j \leqslant c^*|\xi|^2$, $\forall(\xi, x) \in \mathbb{R}^2 \times \overline{\Omega}$, $c_*, c^* > 0$; (A2) ϕ is of class C^2 with respect to the variable y, for any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geqslant \lambda > 0$.

Lemma 1 (Ref. 13) *For every function $g \in L^p(\Omega)$ ($p \geqslant 1$), the solution y of*

$$-\operatorname{div}(A\nabla y) + \phi(y) = g \text{ in } \Omega,\ y|_{\partial\Omega} = 0,$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant C such that

$$\|y\|_{W^{2,p}(\Omega)} \leqslant C\,\|g\|_{L^p(\Omega)}\,.$$

Next, we introduce the co-state elliptic equations

$$\operatorname{div}\mathbf{q} + \phi'(y)z = y - y_{\mathrm{d}},\ \mathbf{q} = -A(\nabla z + \mathbf{p} - \mathbf{p}_{\mathrm{d}}), \quad (4)$$

with boundary condition $z = 0$, $x \in \partial\Omega$. The existence of a unique solution of (2) and (4) is justified by Lemma 1. Furthermore, we make the following realistic assumption (A3): $u \in W^{1,\infty}(\Omega)$, $y, z \in H^3(\Omega)$.

MIXED METHODS FOR OPTIMAL CONTROL PROBLEM

We shall construct a discretized scheme for the nonlinear optimal control problem (1)–(3) by using mixed finite element methods and give its equivalent optimality conditions.

Let $W = L^2(\Omega)$,

$$\mathbf{V} = H(\operatorname{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2, \operatorname{div}\mathbf{v} \in L^2(\Omega)\}.$$

The Hilbert space $\mathbf{V}$ is equipped with the following norm: $\|\mathbf{v}\|_{\mathrm{div}} = \|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = (\|\mathbf{v}\|^2 + \|\operatorname{div}\mathbf{v}\|^2)^{1/2}$. A weak formulation of the optimal control problem (1)–(3) is to find $(\mathbf{p}, y, u) \in \mathbf{V} \times W \times U_{\mathrm{ad}}$ such that

$$\min_{u \in U_{\mathrm{ad}}} \left\{ \tfrac{1}{2}\|\mathbf{p} - \mathbf{p}_{\mathrm{d}}\|^2 + \tfrac{1}{2}\|y - y_{\mathrm{d}}\|^2 + \frac{\alpha}{2}\|u\|^2 \right\} \quad (5)$$

$$(A^{-1}\mathbf{p}, \mathbf{v}) - (y, \operatorname{div}\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in \mathbf{V}, \quad (6)$$

$$(\operatorname{div}\mathbf{p}, w) + (\phi(y), w) = (u, w), \qquad \forall w \in W, \quad (7)$$

where the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ is denoted by $(\cdot,\cdot)$. It is well known[14] that the convex control problem (5)–(7) has a solution $(\mathbf{p}^*, y^*, u^*)$, and that if a triplet $(\mathbf{p}^*, y^*, u^*) \in \mathbf{V} \times W \times U$ is the solution of (5)–(7), then there exists a co-state $(\mathbf{q}^*, z^*) \in \mathbf{V} \times W$ such that $(\mathbf{p}^*, y^*, \mathbf{q}^*, z^*, u^*)$ satisfies the following optimality conditions:

$$(A^{-1}\mathbf{p}^*, \mathbf{v}) - (y^*, \operatorname{div}\mathbf{v}) = 0, \quad (8)$$

$$(\operatorname{div}\mathbf{p}^*, w) + (\phi(y^*), w) = (u^*, w), \quad (9)$$

$$(A^{-1}\mathbf{q}^*, \mathbf{v}) - (z^*, \operatorname{div}\mathbf{v}) = -(\mathbf{p}^* - \mathbf{p}_{\mathrm{d}}, \mathbf{v}), \quad (10)$$

$$(\operatorname{div}\mathbf{q}^*, w) + (\phi'(y^*)z^*, w) = (y^* - y_{\mathrm{d}}, w), \quad (11)$$

$$(z^* + \alpha u^*, \tilde{u} - u^*) \geqslant 0, \quad (12)$$

where $\mathbf{v} \in \mathbf{V}$, $w \in W$ and $\tilde{u} \in U_{\mathrm{ad}}$.

We now introduce the discretized problem by considering a family of triangulations $\mathscr{T}_h$ of $\bar{\Omega}$. With each element $T_i \in \mathscr{T}_h$, we associate two parameters $\rho(T_i)$ and $\sigma(T_i)$, where $\rho(T_i)$ denotes the diameter of the set T_i and $\sigma(T_i)$ is the diameter of the largest ball contained in T_i. The mesh size of the grid is defined by $h = \max_{T_i \in \mathscr{T}_h} \rho(T_i)$. We suppose that the following regularity assumptions are satisfied. There exist two positive constants ϱ_1 and ϱ_2 such that $(\rho(T_i)/\sigma(T_i)) \leqslant \varrho_1$, $(h/\rho(T_i)) \leqslant \varrho_2$ hold for all $T_i \in \mathscr{T}_h$ and all $h > 0$.

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the lowest order Raviart-Thomas mixed finite element space[15], namely,

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \forall T_i \in \mathscr{T}_h, \mathbf{v}|_{T_i} \in P_0^2(T_i) + x \cdot P_0(T_i)\},$$

$$W_h = \{w \in W : \forall T_i \in \mathscr{T}_h, w|_{T_i} \in P_0(T_i)\},$$

where $P_0(T_i)$ indicates a constant on T_i. To approximate the control, we use the following cone of nonnegative piecewise constant functions:

$$U_h = \{\tilde{u} \in U_{\mathrm{ad}} :\ \tilde{u}|_{T_i} = \text{constant}, \quad \forall\ T_i \in \mathscr{T}_h\}.$$

Then we introduce the following Raviart-Thomas projection[16]:

$$\Pi_h \times P_h : \mathbf{V} \times W \longrightarrow \mathbf{V}_h \times W_h,$$

which has the following properties.
(i) P_h is the local $L^2(\Omega)$ projection.
(ii) Π_h and P_h satisfy

$$\operatorname{div} \circ \Pi_h = P_h \circ \operatorname{div}. \quad (13)$$

Using property (i) and (13), we can obtain

$$(\operatorname{div}(\mathbf{v} - \Pi_h\mathbf{v}), w_h) = 0,\ w_h \in W_h,$$

$$(\operatorname{div}\mathbf{v}_h, w - P_h w) = 0,\ \mathbf{v}_h \in \mathbf{V}_h.$$

(iii) The following approximate properties hold[17]:

$$\|\mathbf{v}-\Pi_h\mathbf{v}\|_{0,\rho} \leqslant Ch^r \|\mathbf{v}\|_{r,\rho}, \quad \frac{1}{\rho} < r \leqslant k+1, \tag{14}$$

$$\|\operatorname{div}(\mathbf{v}-\Pi_h\mathbf{v})\|_{-t} \leqslant Ch^{r+t} \|\operatorname{div}\mathbf{v}\|_r, \tag{15}$$

$$0 \leqslant r, t \leqslant k+1, \tag{16}$$

$$\|w-P_h w\|_{-t,\rho} \leqslant Ch^{r+t} \|w\|_{r,\rho}, \tag{17}$$

$$0 \leqslant r, t \leqslant k+1, \tag{18}$$

where $\|.\|_{r,\rho}$ denotes the norm of the usual Sobolev space $W^{r,\rho}(\Omega)$ for $1 \leqslant \rho \leqslant +\infty$ and $r \geqslant 0$.

The mixed finite element approximation of (5)–(7) is to find $(\mathbf{p}_h, y_h, u_h) \in \mathbf{V}_h \times W_h \times U_h$ such that

$$\min_{u\in U_h} \tfrac{1}{2}\{\|\mathbf{p}_h-\mathbf{p}_\mathrm{d}\|^2 + \|y_h-y_\mathrm{d}\|^2 + \alpha\|u_h\|^2\} \tag{19}$$

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div}\mathbf{v}_h) = 0, \tag{20}$$

$$(\operatorname{div}\mathbf{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \tag{21}$$

where $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$.

The optimal control problem (19)–(21) again has a solution $(\mathbf{p}_h^*, y_h^*, u_h^*)$, and that if a triplet $(\mathbf{p}_h^*, y_h^*, u_h^*) \in \mathbf{V}_h \times W_h \times U_h$ is the solution of (19)–(21), then there is a co-state $(\mathbf{q}_h^*, z_h^*) \in \mathbf{V}_h \times W_h$ such that $(\mathbf{p}_h^*, y_h^*, \mathbf{q}_h^*, z_h^*, u_h^*)$ satisfies the following discretized optimality conditions:

$$(A^{-1}\mathbf{p}_h^*, \mathbf{v}_h) - (y_h^*, \operatorname{div}\mathbf{v}_h) = 0, \tag{22}$$

$$(\operatorname{div}\mathbf{p}_h^*, w_h) + (\phi(y_h^*), w_h) = (u_h^*, w_h), \tag{23}$$

$$(A^{-1}\mathbf{q}_h^*, \mathbf{v}_h) - (z_h^*, \operatorname{div}\mathbf{v}_h) = -(\mathbf{p}_h^* - \mathbf{p}_\mathrm{d}, \mathbf{v}_h), \tag{24}$$

$$(\operatorname{div}\mathbf{q}_h^*, w_h) + (\phi'(y_h^*)z_h^*, w_h) = (y_h^* - y_\mathrm{d}, w_h), \tag{25}$$

$$(z_h^* + \alpha u_h^*, \tilde{u}_h - u_h^*) \geqslant 0, \tag{26}$$

where $\mathbf{v}_h \in \mathbf{V}_h$, $w_h \in W_h$ and $\tilde{u}_h \in U_h$.

We now shall use some intermediate variables. For any control function $\tilde{u} \in U_\mathrm{ad}$, we define the state solution $(\mathbf{p}^*(\tilde{u}), y^*(\tilde{u}), \mathbf{q}^*(\tilde{u}), z^*(\tilde{u}))$ associated with $\tilde{u}$ which satisfies

$$(A^{-1}\mathbf{p}^*(\tilde{u}), \mathbf{v}) - (y^*(\tilde{u}), \operatorname{div}\mathbf{v}) = 0, \tag{27}$$

$$(\operatorname{div}\mathbf{p}^*(\tilde{u}), w) + (\phi(y^*(\tilde{u})), w) = (\tilde{u}, w), \tag{28}$$

$$(A^{-1}\mathbf{q}^*(\tilde{u}), \mathbf{v}) - (z^*(\tilde{u}), \operatorname{div}\mathbf{v}) = -(\mathbf{p}^*(\tilde{u}) - \mathbf{p}_\mathrm{d}, \mathbf{v}), \tag{29}$$

$$(\operatorname{div}\mathbf{q}^*(\tilde{u}), w) + (\phi'(y^*(\tilde{u}))z^*(\tilde{u}), w) = (y^*(\tilde{u}) - y_\mathrm{d}, w), \tag{30}$$

where $\mathbf{v} \in \mathbf{V}$ and $w \in W$. We define the discrete state solution $(\mathbf{p}_h^*(\tilde{u}), y_h^*(\tilde{u}), \mathbf{q}_h^*(\tilde{u}), z_h^*(\tilde{u}))$ corresponding to $\tilde{u}$ which satisfies

$$(A^{-1}\mathbf{p}_h^*(\tilde{u}), \mathbf{v}_h) - (y_h^*(\tilde{u}), \operatorname{div}\mathbf{v}_h) = 0, \tag{31}$$

$$(\operatorname{div}\mathbf{p}_h^*(\tilde{u}), w_h) + (\phi(y_h^*(\tilde{u})), w_h) = (\tilde{u}, w_h), \tag{32}$$

$$(A^{-1}\mathbf{q}_h^*(\tilde{u}), \mathbf{v}_h) - (z_h^*(\tilde{u}), \operatorname{div}\mathbf{v}_h) = -(\mathbf{p}_h^*(\tilde{u}) - \mathbf{p}_\mathrm{d}, \mathbf{v}_h), \tag{33}$$

$$(\operatorname{div}\mathbf{q}_h^*(\tilde{u}), w_h) + (\phi'(y_h^*(\tilde{u}))z_h^*(\tilde{u}), w_h) = (y_h^*(\tilde{u}) - y_\mathrm{d}, w_h), \tag{34}$$

where $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$. With these definitions, the exact state solution and its approximation can be written as

$$(\mathbf{p}^*, y^*, \mathbf{q}^*, z^*) = (\mathbf{p}^*(u^*), y^*(u^*), \mathbf{q}^*(u^*), z^*(u^*)),$$
$$(\mathbf{p}_h^*, y_h^*, \mathbf{q}_h^*, z_h^*) = (\mathbf{p}_h^*(u_h^*), y_h^*(u_h^*), \mathbf{q}_h^*(u_h^*), z_h^*(u_h^*)).$$

For $\varphi \in W_h$, we shall write[18]

$$\phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2, \tag{35}$$

where $\tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi + t(\rho - \varphi))\mathrm{d}t$, $\tilde{\phi}''(\varphi) = \int_0^1 (1-t)\phi''(\rho + t(\varphi - \rho))\mathrm{d}t$ are bounded functions in $\bar{\Omega}$.

SUPERCONVERGENCE

Firstly, we can obtain the following technical results[19]:

Lemma 2 *Suppose* **(A1)** *hold. Let $\gamma \in C^1(\Omega)$, $\omega \in \mathbf{V}$, $\varphi \in L^2(\Omega)^2$, and $\psi \in L^2(\Omega)$. If $\tau \in W_h$ satisfies*

$$(A^{-1}\omega, \mathbf{v}_h) - (\tau, \operatorname{div}\mathbf{v}_h) = (\varphi, \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\operatorname{div}\omega, w_h) + (\gamma\tau, w_h) = (\psi, w_h), \quad \forall\ w_h \in W_h,$$

then there exists a constant C such that

$$\|\tau\|_0 \leqslant C\big(h\|\omega\|_0 + h^2\|\operatorname{div}\omega\|_0 + \|\varphi\|_0 + \|\psi\|_0\big), \tag{36}$$

for h sufficiently small.

For any $\tilde{u} \in U$, let

$$\varepsilon_1 := \mathbf{p}^*(\tilde{u}) - \mathbf{p}_h^*(\tilde{u}), \quad e_1 := y^*(\tilde{u}) - y_h^*(\tilde{u}).$$

To analyse the intermediate errors, let us first note the following error equations from (22)–(23) and (27)–(28):

$$(A^{-1}\varepsilon_1, \mathbf{v}_h) - (e_1, \operatorname{div}\mathbf{v}_h) = 0, \tag{37}$$

$$(\operatorname{div}\varepsilon_1, w_h) + (\tilde{\phi}'(y^*(\tilde{u}))e_1, w_h) = 0, \tag{38}$$

where $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$.

By using Lemma 2, we can establish the following error estimates:

Lemma 3 *Suppose that assumptions (A1–3) are fulfilled. Let $y^*(\tilde{u})$ and $y_h^*(\tilde{u})$ be the solutions of (27)–(30) and (31)–(34), respectively. If the intermediate solutions satisfy $\mathbf{p}^*(\tilde{u}) \in [H^2(\Omega)]^2$, $y^*(\tilde{u}) \in H^4(\Omega)$ then we have*

$$\left\|y_h^*(\tilde{u}) - y^*(\tilde{u})\right\| + \left\|\mathbf{p}_h^*(\tilde{u}) - \mathbf{p}^*(\tilde{u})\right\|_{\text{div}} \leqslant Ch^2. \tag{39}$$

Lemma 4 *Suppose that assumptions (A1–3) are fulfilled. Let $z^*(\tilde{u})$ and $z_h^*(\tilde{u})$ be the solutions of (27)–(30) and (31)–(34), respectively. If the intermediate solutions satisfy $\mathbf{p}^*(\tilde{u}), \mathbf{q}^*(\tilde{u}) \in [H^2(\Omega)]^2$, $y^*(\tilde{u}), z^*(\tilde{u}) \in H^4(\Omega)$ then*

$$\left\|z_h^*(\tilde{u}) - z^*(\tilde{u})\right\| + \left\|\mathbf{q}_h^*(\tilde{u}) - \mathbf{q}^*(\tilde{u})\right\|_{\text{div}} \leqslant Ch^2.$$

Lemma 5 *Suppose that assumptions (A1–3) are valid. Let $P_h u^*$ be the local $L^2(\Omega)$ projection of the exact control u^* and $z^*(P_h u^*)$ and $z^*(u^*)$ be the solutions of (27)–(30) with $\tilde{u} = P_h u^*$ and $\tilde{u} = u^*$, respectively. Then we have*

$$\|z^*(P_h u^*) - z^*(u^*)\| \leqslant Ch^2. \tag{40}$$

Let $(\mathbf{p}^*(u^*), y^*(u^*))$ and $(\mathbf{p}_h^*(u_h^*), y_h^*(u_h^*))$ be the solutions of (8)–(12) and (22)–(26), respectively. Let $J(\cdot) : U \to \mathbb{R}$ be a G-differential convex functional with the following form:

$$J(u^*) = \tfrac{1}{2}\|\mathbf{p}^* - \mathbf{p}_\text{d}\|^2 + \tfrac{1}{2}\|y^* - y_\text{d}\|^2 + \frac{\alpha}{2}\|u^*\|^2.$$

It can be shown that

$$(J'(u^*), v) = (z^* + \alpha u^*, v),$$
$$(J'(u_h^*), v) = (z^*(u_h^*) + \alpha u_h^*, v).$$

In many applications, $J(\cdot)$ is uniform convex near the solution u^* (see Ref. 5). Then there is a $c > 0$, independent of h, such that

$$(J'(u^*) - J'(u_h^*), u^* - u_h^*) \geqslant c\left\|u^* - u_h^*\right\|^2, \tag{41}$$

where u^* and u_h^* are the solutions of (12) and (26), respectively. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the optimal control problem, which are assumed in many studies on numerical methods of the problem.

Let

$$\Omega^+ = \{\cup T_i :\ T_i \in \Omega,\ u^*|_{T_i} > 0\},$$
$$\Omega^0 = \{\cup T_i :\ T_i \in \Omega,\ u^*|_{T_i} = 0\},$$
$$\Omega^\text{b} = \Omega \setminus (\Omega^+ \cup \Omega^0).$$

We will assume that u^* and $\mathscr{T}_h$ are regular such that $|\Omega^\text{b}| \leqslant Ch$. We are now able to obtain our first main result.

Theorem 1 *Suppose that assumptions (A1–3) are satisfied. Let $P_h u^*$ be the local $L^2(\Omega)$ projection of the exact control u^* and u_h^* be the solution of (22)–(26). Then we have the estimate*

$$\left\|P_h u^* - u_h^*\right\| \leqslant Ch^{3/2}. \tag{42}$$

Proof: Let $\mathbf{v} = u_h^*$ in (12) and $\mathbf{v}_h = P_h u^*$ in (26). We have

$$(z^* + \alpha u^*, u_h^* - u^*) \geqslant 0,$$
$$(z_h^* + \alpha u_h^*, P_h u^* - u_h^*) \geqslant 0.$$

Adding the two inequalities gives

$$(z_h^* + \alpha u_h^* - z^* - \alpha u^*, P_h u^* - u_h^*) + (z^* + \alpha u^*, P_h u^* - u^*) \geqslant 0.$$

So we obtain

$$\begin{aligned}\alpha(P_h u^* - u_h^*, P_h u^* - u_h^*) &= \alpha(u^* - u_h^*, P_h u^* - u_h^*)\\ &\leqslant (z_h^* - z^*, P_h u^* - u_h^*)\\ &\quad + (z^* + \alpha u^*, P_h u^* - u^*).\end{aligned} \tag{43}$$

Clearly,

$$\begin{aligned}(z_h^* - z^*, P_h u^* - u_h^*) &= (z_h^* - z^*(u_h^*), P_h u^* - u_h^*)\\ &\quad + (z^*(u_h^*) - z^*(P_h u^*), P_h u^* - u_h^*)\\ &\quad + (z^*(P_h u^*) - z^*(u^*), P_h u^* - u_h^*).\end{aligned} \tag{44}$$

Then

$$\begin{aligned}&\alpha(P_h u^* - u_h^*, P_h u^* - u_h^*)\\ &\quad - (z^*(u_h^*) - z^*(P_h u^*), P_h u^* - u_h^*)\\ &\leqslant (z_h^* - z^*(u_h^*), P_h u^* - u_h^*)\\ &\quad + (z^*(P_h u^*) - z^*(u^*), P_h u^* - u_h^*)\\ &\quad + (z^* + \alpha u^*, P_h u^* - u^*)\\ &\equiv E_1 + E_2 + E_3.\end{aligned} \tag{45}$$

Now we find bounds for the E_i. From Lemma 4, we have

$$\begin{aligned}E_1 &= (z_h^* - z^*(u_h^*), P_h u^* - u_h^*)\\ &\leqslant C\left\|z_h^* - z^*(u_h^*)\right\| \cdot \left\|P_h u^* - u_h^*\right\|\\ &\leqslant Ch^2\left\|P_h u^* - u_h^*\right\|.\end{aligned} \tag{46}$$

From (40),

$$\begin{aligned}E_2 &= (z^*(P_h u^*) - z^*(u^*), P_h u^* - u_h^*)\\ &\leqslant C\|z^*(P_h u^*) - z^*(u^*)\| \cdot \left\|P_h u^* - u_h^*\right\|\\ &\leqslant Ch^2\left\|P_h u^* - u_h^*\right\|.\end{aligned} \tag{47}$$

Finally,

$$\begin{aligned} E_3 &= (z^* + \alpha u^*, P_h u^* - u^*) \\ &= \int_{\Omega^+} (z^* + \alpha u^*, P_h u^* - u^*)\,\mathrm{d}x \\ &\quad + \int_{\Omega^0} (z^* + \alpha u^*, P_h u^* - u^*)\,\mathrm{d}x \\ &\quad + \int_{\Omega^b} (z^* + \alpha u^*, P_h u^* - u^*)\,\mathrm{d}x. \end{aligned} \tag{48}$$

From the definition of Ω^0 we note that $(P_h u^* - u^*)|_{\Omega^0} = 0$. It is clear that

$$\int_{\Omega^0} (z^* + \alpha u^*, P_h u^* - u^*)\,\mathrm{d}x = 0. \tag{49}$$

From (12), we have pointwise a.e. $z^* + \alpha u^* \geqslant 0$. We choose $\tilde{u}|_{\Omega^+} = 0$ and $\tilde{u}|_{\Omega\setminus\Omega^+} = u^*$ so that $(z^* + \alpha u^*, u^*)|_{\Omega^+} \leqslant 0$. Hence $(z^* + \alpha u^*)|_{\Omega^+} = 0$. Let $\pi^c u^*$ be the integral operator such that $\pi^c u^*|_{T_i} = \int_{T_i} u^* / \int_{T_i} 1$. It follows from the definition of π^c that

$$\begin{aligned} &(z^* + \alpha u^*, P_h u^* - u^*) \\ &\quad = (z^* + \alpha u^*, P_h u^* - u^*)_{\Omega^b} \\ &\quad \leqslant (z^* + \alpha u^* - \pi^c(z^* + \alpha u^*), P_h u^* - u^*)_{\Omega^b} \\ &\quad \leqslant Ch^2 \|z^* + \alpha u^*\|_{1,\Omega^b} \|u^*\|_{1,\Omega^b} \\ &\quad \leqslant Ch^2 \|z^* + \alpha u^*\|_{1,\infty} \|u^*\|_{1,\infty} \cdot \left|\Omega^b\right| \\ &\quad \leqslant Ch^3. \end{aligned} \tag{50}$$

Then it follows from assumption (41), (45)–(50), and the Schwartz inequality that

$$\begin{aligned} &c\,\|P_h u^* - u^*\|_0^2 \\ &\quad \leqslant (J'(P_h u^*) - J'(u_h^*), P_h u^* - u^*) \\ &\quad = \alpha(P_h u^* - u_h^*, P_h u^* - u_h^*) \\ &\qquad - (z^*(u_h^*) - z^*(P_h u^*), P_h u^* - u_h^*) \\ &\quad \leqslant Ch^3 + Ch^2 \left\|P_h u^* - u_h^*\right\| \\ &\quad \leqslant Ch^3 + \delta \left\|P_h u^* - u_h^*\right\|^2. \end{aligned} \tag{51}$$

The estimate (42) follows from taking $\delta = \frac{1}{2}c$. □

Next, we establish the following superconvergence result for state and co-state.

Theorem 2 *Suppose that assumptions (A1–3) are satisfied. Let* $(\mathbf{p}^*, y^*, \mathbf{q}^*, z^*, u^*) \in (\mathbf{V}\times W)^2 \times U_{\mathrm{ad}}$ *be the solutions defined in* (8)–(12) *and* $(\mathbf{p}_h^*, y_h^*, \mathbf{q}_h^*, z_h^*, u_h^*) \in (\mathbf{V}_h \times W_h)^2 \times U_h$ *be the solutions of* (22)–(26). *Then we have*

$$\left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\|_{\mathrm{div}} + \left\|P_h y^* - y_h^*\right\| \leqslant Ch^{3/2}, \tag{52}$$

$$\left\|\Pi_h \mathbf{q}^* - \mathbf{q}_h^*\right\|_{\mathrm{div}} + \left\|P_h z^* - z_h^*\right\| \leqslant Ch^{3/2}. \tag{53}$$

Proof: It follows from (8)–(12) and (22)–(26) that we have the error equations:

$$\begin{aligned} &(A^{-1}(\mathbf{p}^* - \mathbf{p}_h^*), \mathbf{v}_h) - (y^* - y_h^*, \mathrm{div}\,\mathbf{v}_h) = 0, \\ &(\mathrm{div}(\mathbf{p}^* - \mathbf{p}_h^*), w_h) + (\tilde{\phi}'(y^*)(y^* - y_h^*), w_h) \\ &\quad = (u^* - u_h^*, w_h), \\ &(A^{-1}(\mathbf{q}^* - \mathbf{q}_h^*), \mathbf{v}_h) - (z^* - z_h^*, \mathrm{div}\,\mathbf{v}_h) \\ &\quad = -(\mathbf{p}^* - \mathbf{p}_h^*, \mathbf{v}_h), \\ &(\mathrm{div}(\mathbf{q}^* - \mathbf{q}_h^*), w_h) + (\phi'(y^*)(z^* - z_h^*), w_h) \\ &\quad = (y^* - y_h^*, w_h) - (\tilde{\phi}''(y^*)(y^* - y_h^*) z_h^*, w_h), \end{aligned}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$. By using the definitions of projections Π_h and P_h, the above equations can be rewritten as

$$\begin{aligned} &(A^{-1}(\Pi_h \mathbf{p}^* - \mathbf{p}_h^*), \mathbf{v}_h) - (P_h y^* - y_h^*, \mathrm{div}\,\mathbf{v}_h) \\ &\quad = \phi_1(\mathbf{v}_h), \end{aligned} \tag{54}$$

$$\begin{aligned} &(\mathrm{div}(\Pi_h \mathbf{p}^* - \mathbf{p}_h^*), w_h) \\ &\quad + (\tilde{\phi}'(y^*)(P_h y^* - y_h^*), w_h) = \psi_1(w_h), \end{aligned} \tag{55}$$

$$\begin{aligned} &(A^{-1}(\Pi_h \mathbf{q}^* - \mathbf{q}_h^*), \mathbf{v}_h) \\ &\quad - (P_h z^* - z_h^*, \mathrm{div}\,\mathbf{v}_h) = \phi_2(\mathbf{v}_h), \end{aligned} \tag{56}$$

$$\begin{aligned} &(\mathrm{div}(\Pi_h \mathbf{q}^* - \mathbf{q}_h^*), w_h) \\ &\quad + (\phi'(y^*)(P_h z^* - z_h^*), w_h) = \psi_2(w_h), \end{aligned} \tag{57}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$, where

$$\begin{aligned} \phi_1(\mathbf{v}_h) &= -(A^{-1}(\mathbf{p}^* - \Pi_h \mathbf{p}^*), \mathbf{v}_h), \\ \psi_1(w_h) &= (u^* - u_h^*, w_h) - (\tilde{\phi}'(y^*)(y^* - P_h y^*), w_h), \\ \phi_2(\mathbf{v}_h) &= -(\mathbf{p}^* - \mathbf{p}_h^*, \mathbf{v}_h) - (A^{-1}(\mathbf{q}^* - \Pi_h \mathbf{q}^*), \mathbf{v}_h), \\ \psi_2(w_h) &= (y^* - y_h^*, w_h) - (\phi'(y^*)(z^* - P_h z^*), w_h) \\ &\quad - (\tilde{\phi}''(y^*)(y^* - y_h^*) z_h^*, w_h). \end{aligned}$$

since the terms $\phi_1(\mathbf{v}_h), \psi_1(w_h), \phi_2(\mathbf{v}_h), \psi_2(w_h)$ can be regarded as linear functionals of $\mathbf{v}_h$ and w_h defined on $\mathbf{V}_h$ and W_h, respectively. Then we know from the stability result[19,20] that

$$\begin{aligned} &\left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\|_{\mathrm{div}} + \left\|P_h y^* - y_h^*\right\| \\ &\quad \leqslant C\left\{\sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{|\phi_1(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\mathrm{div}}} + \sup_{w_h\in W_h} \frac{|\psi_1(w_h)|}{\|w_h\|}\right\}, \\ &\left\|\Pi_h \mathbf{q}^* - \mathbf{q}_h^*\right\|_{\mathrm{div}} + \left\|P_h z^* - z_h^*\right\| \\ &\quad \leqslant C\left\{\sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{|\phi_2(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\mathrm{div}}} + \sup_{w_h\in W_h} \frac{|\psi_2(w_h)|}{\|w_h\|}\right\}. \end{aligned}$$

It is easy to see that

$$(\mathbf{p}^* - \mathbf{p}_h^*, \mathbf{v}_h) = (\mathbf{p}^* - \Pi_h \mathbf{p}^*, \mathbf{v}_h) + (\Pi_h \mathbf{p}^* - \mathbf{p}_h^*, \mathbf{v}_h), \quad (58)$$

$$(y^* - y_h^*, w_h) = (y^* - P_h y^*, w_h) + (P_h y^* - y_h^*, w_h) = (P_h y^* - y_h^*, w_h). \quad (59)$$

By the standard superconvergence of mixed finite element methods[21–23], we have

$$(\tilde{\phi}'(y^*)(y^* - P_h y^*), w_h) \leqslant Ch^2 \|y^*\|_{H^1(\Omega)} \|w_h\|, \quad (60)$$

$$(\phi'(y^*)(z^* - P_h z^*), w_h) \leqslant Ch^2 \|z^*\|_{H^1(\Omega)} \|w_h\|, \quad (61)$$

$$(\tilde{\phi}''(y^*)(y^* - P_h y^*) z_h^*, w_h) \leqslant Ch^2 \|y^*\|_{H^1(\Omega)} \|w_h\|. \quad (62)$$

Here we only give the proof of (60). By using the definition of the local $L^2(\Omega)$ projection P_h, we obtain

$$(P_h(\tilde{\phi}'(y^*))(y^* - P_h y^*), w_h) = (y^* - P_h y^*, P_h(\tilde{\phi}'(y^*)) w_h) = 0. \quad (63)$$

Then

$$\begin{aligned}(\tilde{\phi}'(y^*)(y^* - P_h y^*), w_h) &= ((\tilde{\phi}'(y^*) - P_h(\tilde{\phi}'(y^*)))(y^* - P_h y^*), w_h) \\ &\leqslant Ch \|\phi\|_{2,\infty} \|y^* - P_h y^*\| \|w_h\| \\ &\leqslant Ch^2 \|y^*\|_{H^1(\Omega)} \|w_h\|. \end{aligned} \quad (64)$$

Under the condition y^*, $z^* \in H^3(\Omega)$, applying the integral identity technique[24], we see that

$$(A^{-1}(\mathbf{p}^* - \Pi_h \mathbf{p}^*), \mathbf{v}_h) \leqslant Ch^2 \|y^*\|_{H^3(\Omega)} \|\mathbf{v}_h\|, \quad (65)$$

$$(A^{-1}(\mathbf{q}^* - \Pi_h \mathbf{q}^*), \mathbf{v}_h) \leqslant Ch^2 \|z^*\|_{H^3(\Omega)} \|\mathbf{v}_h\|, \quad (66)$$

$$(\mathbf{p}^* - \Pi_h \mathbf{p}^*, \mathbf{v}_h) \leqslant Ch^2 \|y^*\|_{H^3(\Omega)} \|\mathbf{v}_h\|. \quad (67)$$

By adding (54) and (55) to $\mathbf{v}_h = \Pi_h \mathbf{p}^* - \mathbf{p}_h^*$ and $w_h = P_h y^* - y_h^*$, we have

$$\begin{aligned}&(A^{-1}(\Pi_h \mathbf{p}^* - \mathbf{p}_h^*), \Pi_h \mathbf{p}^* - \mathbf{p}_h^*) + (\tilde{\phi}'(y^*)(P_h y^* - y_h^*), P_h y^* - y_h^*) \\ &= (u^* - u_h^*, P_h y^* - y_h^*) - (\tilde{\phi}'(y^*)(y^* - P_h y^*), P_h y^* - y_h^*) \\ &\quad - (A^{-1}(\mathbf{p}^* - \Pi_h \mathbf{p}^*), \Pi_h \mathbf{p}^* - \mathbf{p}_h^*).\end{aligned}$$

Table 1 The errors of Example 1 on sequential uniform refined meshes.

resolution	$\|u^* - u_h^*\|$	$\|P_h u^* - u_h^*\|$
16×16	1.133×10^{-1}	2.112×10^{-2}
32×32	5.668×10^{-2}	7.168×10^{-3}
64×64	2.834×10^{-2}	2.138×10^{-3}
128×128	1.417×10^{-2}	6.948×10^{-4}

Using the assumption of $A(x)$, ϕ and Hölder's inequality, for any small $\delta > 0$, we obtain

$$\begin{aligned}&\frac{1}{c^*} \left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\|^2 + \lambda \left\|P_h y^* - y_h^*\right\|^2 \\ &\leqslant (A^{-1}(\Pi_h \mathbf{p}^* - \mathbf{p}_h^*), \Pi_h \mathbf{p}^* - \mathbf{p}_h^*) \\ &\quad + (\tilde{\phi}'(y^*)(P_h y^* - y_h^*), P_h y^* - y_h^*) \\ &= (u^* - u_h^*, P_h y^* - y_h^*) \\ &\quad - (\tilde{\phi}'(y^*)(y^* - P_h y^*), P_h y^* - y_h^*) \\ &\quad - (A^{-1}(\mathbf{p}^* - \Pi_h \mathbf{p}^*), \Pi_h \mathbf{p}^* - \mathbf{p}_h^*) \\ &\leqslant \left\|u^* - u_h^*\right\| \cdot \left\|P_h y^* - y_h^*\right\| \\ &\quad + \|\phi\|_{2,\infty} \|y^* - P_h y^*\| \cdot \left\|P_h y^* - y_h^*\right\| \\ &\quad + C \|\mathbf{p}^* - \Pi_h \mathbf{p}^*\| \cdot \left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\| \\ &\leqslant \left\|u^* - u_h^*\right\| \cdot \left\|P_h y^* - y_h^*\right\| \\ &\quad + Ch^2 \|y^*\|_{H^2(\Omega)} \cdot \left\|P_h y^* - y_h^*\right\| \\ &\quad + Ch^2 \|y^*\|_{H^3(\Omega)} \cdot \left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\| \\ &\leqslant Ch^4 + C \left\|u^* - u_h^*\right\|^2 \\ &\quad + \delta(\left\|P_h y^* - y_h^*\right\|^2 + \left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\|^2).\end{aligned}$$

The formula is equivalent to

$$\left\|\Pi_h \mathbf{p}^* - \mathbf{p}_h^*\right\| + \left\|P_h y^* - y_h^*\right\| \leqslant Ch^2 + C \left\|u^* - u_h^*\right\|.$$

Note that

$$(u^* - u_h^*, w_h) = (u^* - P_h u^*, w_h) + (P_h u^* - u_h^*, w_h).$$

It is easy to see that $(u^* - P_h u^*, w_h) = 0$. By using Theorem 1, we clearly see that

$$\begin{aligned}(P_h u^* - u_h^*, w_h) &\leqslant \left\|P_h u^* - u_h^*\right\| \cdot \|w_h\| \\ &\leqslant Ch^{3/2} \|w_h\|.\end{aligned}$$

From the above analysis, we can obtain (52) and (53). □

NUMERICAL EXAMPLES

We present two examples to test the superconvergence results of the control. The optimization problems were solved numerical by projected

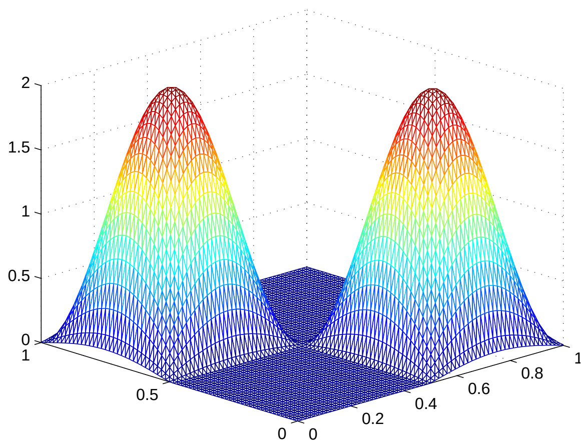

Fig. 1 The profile of the numerical solution of Example 1.

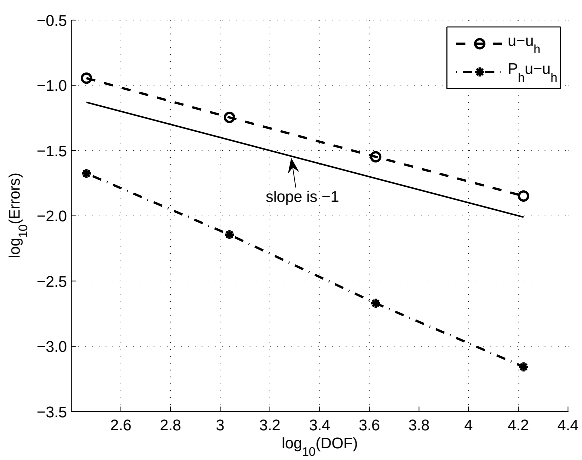

Fig. 2 Convergence orders of $u^* - u_h^*$ and $P_h u^* - u_h^*$.

gradient methods, with codes developed based on AFEPACK[25]. The control function u is discretized by piecewise constant functions, where the state $(y, \mathbf{p})$ and the co-state $(z, \mathbf{q})$ were approximated by the lowest order Raviart-Thomas mixed finite element functions. In the two examples, we choose the domain $\Omega = [0,1] \times [0,1]$.

Example 1 We consider the two-dimensional nonlinear elliptic optimal control problem

$$\min_{u \in U_{\text{ad}}} \tfrac{1}{2}\{\|\mathbf{p} - \mathbf{p}_\text{d}\|^2 + \|y - y_\text{d}\|^2 + \|u\|^2\}$$

subject to the state equation

$$\operatorname{div}\mathbf{p} + y^3 = u + f,\ \mathbf{p} = -\nabla y, \quad x \in \Omega,$$

with the boundary condition $y = 0$, $x \in \partial\Omega$, and the admissible set $U_\text{ad} = \{u \in L^2(\Omega) :\ u \geqslant 0\}$. Next, we introduce the co-state elliptic equation $\operatorname{div}\mathbf{q} + 3y^2 z = y - y_\text{d}$, $\mathbf{q} = -(\nabla z + \mathbf{p} - \mathbf{p}_\text{d})$, $x \in \Omega$, with the boundary condition $z = 0$, $x \in \partial\Omega$. We choose

$$\begin{aligned}
u &= \max(0, -z),\\
f &= 8\pi^2 y + y^3 - u,\\
y_\text{d} &= (1 - 16\pi^2)y - 3y^2 z,\\
y &= \sin(2\pi x_1)\sin(2\pi x_2),\\
z &= 2\sin(2\pi x_1)\sin(2\pi x_2),\\
\mathbf{p} &= -2\pi(\cos 2\pi x_1 \sin 2\pi x_2, \cos 2\pi x_2 \sin 2\pi x_1),\\
\mathbf{q} &= -\pi(\cos 2\pi x_1 \sin 2\pi x_2, \cos 2\pi x_2 \sin 2\pi x_1),\\
\mathbf{p}_\text{d} &= \pi(\cos 2\pi x_1 \sin 2\pi x_2, \cos 2\pi x_2 \sin 2\pi x_1).
\end{aligned}$$

In the numerical implementation, the profile of the numerical solution is shown in Fig. 1 and the errors $\|u^* - u_h^*\|$ and $\|P_h u^* - u_h^*\|$ obtained on a sequence of uniformly refined meshes are presented in Table 1. The convergence orders on triangle mesh grids are depicted in Fig. 2. It is clear that $\|P_h u^* - u_h^*\|$ has a superconvergence of $O(h^{3/2})$.

Example 2 We consider the following nonlinear optimal control problem:

$$\begin{aligned}
&\min_{u \in U_{\text{ad}}} \tfrac{1}{2}\{\|\mathbf{p} - \mathbf{p}_\text{d}\|^2 + \|y - y_\text{d}\|^2 + \|u\|^2\},\\
&\qquad \operatorname{div}\mathbf{p} + y^7 = u + f,\ \mathbf{p} = -\nabla y,\ x \in \Omega,\\
&\qquad\qquad\qquad\qquad y = 0, \quad x \in \partial\Omega,
\end{aligned}$$

and we introduce co-state elliptic equation $\operatorname{div}\mathbf{q} + 7y^6 z = y - y_\text{d}$, $\mathbf{q} = -(\nabla z + \mathbf{p} - \mathbf{p}_\text{d})$, $x \in \Omega$, with the boundary condition $z = 0$, $x \in \partial\Omega$. We choose that

$$\begin{aligned}
u &= \max(-z, 0), \qquad f = \operatorname{div}\mathbf{p} + y^7 - u,\\
y_\text{d} &= y - \operatorname{div}\mathbf{q} - 7y^6 z, \qquad \mathbf{q} = -\nabla z - \mathbf{p} + \mathbf{p}_\text{d},\\
y &= 2x_1 x_2^2 (1 - x_1^3)(1 - x_2)^2 \sin(8\pi x_1),\\
z &= -x_1 x_2^2 (1 - x_1^3)(1 - x_2)^2 \sin(8\pi x_1),\\
\mathbf{p} &= -\nabla y, \qquad \mathbf{p}_\text{d} = \mathbf{p} + \mathbf{q} + \nabla z.
\end{aligned}$$

The profile of the numerical solution is presented in Fig. 3. The superconvergence behaviour of $\|P_h u^* - u_h^*\|$ is illustrated in Table 2 and Fig. 4.

Table 2 The errors of Example 2 on sequential uniform refined meshes.

resolution	$\|u^* - u_h^*\|$	$\|P_h u^* - u_h^*\|$
16×16	1.189×10^{-3}	2.621×10^{-4}
32×32	5.843×10^{-4}	8.932×10^{-5}
64×64	2.891×10^{-4}	3.129×10^{-5}
128×128	1.438×10^{-4}	1.109×10^{-5}

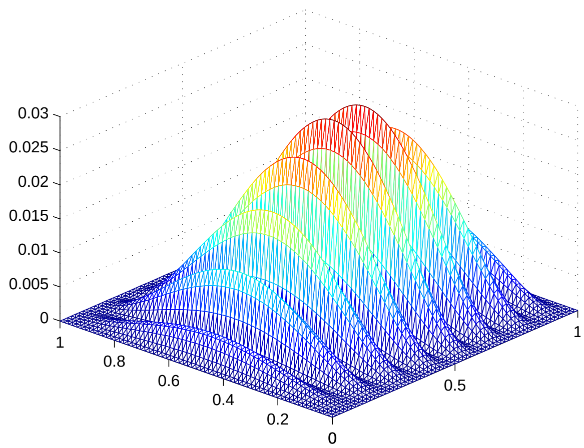

Fig. 3 The profile of the numerical solution of Example 2.

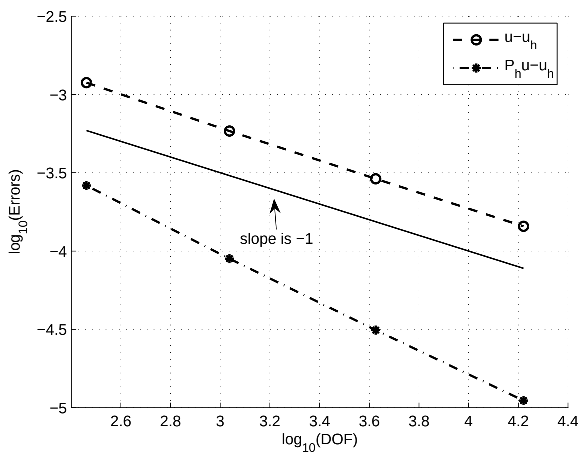

Fig. 4 Convergence orders of $u^* - u_h^*$ and $P_h u^* - u_h^*$.

From the numerical results of the examples, the superconvergence phenomenon can be observed clearly.

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