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On the complete convergence of weighted sums for an array of rowwise negatively superadditive dependent random variables

Xinghui Wang, Xiaoqin Li, Shuhe Hu*

Department of Statistics, Anhui University, Hefei 230601, China

*Corresponding author, e-mail: hushuhe@263.net

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ABSTRACT: In this paper, the complete convergence and the complete moment convergence of weighted sums for an array of negatively superadditive dependent random variables are established. The results generalize the Baum-Katz theorem on negatively superadditive dependent random variables. In particular, the Marcinkiewicz-Zygmund type strong law of large numbers of weights sums for sequences of negatively superadditive dependent random variables is obtained.

KEYWORDS: Baum-Katz type theorem, Marcinkiewicz-Zygmund type strong law of large numbers

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INTRODUCTION

Assume that $\{X_n, n \geqslant 1\}$ is a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{A}, P)$. We first recall the definition of negative association introduced by Block et al[1] and studied in Ref. 2.

Definition 1 A finite family $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$

$$\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leqslant 0$$

whenever f and g are coordinatewise nondecreasing functions such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. An array of random variables $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ is called an array of rowwise NA random variables if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of NA random variables.

Definition 2 [Kemperman[3]] A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called *superadditive* if $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geqslant \phi(\mathbf{x}) + \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

Definition 3 [Hu[4]] A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be *negatively superadditive dependent* (NSD) if

$$E\phi(X_1, X_2, \ldots, X_n) \leqslant E\phi(Y_1, Y_2, \ldots, Y_n), \qquad (1)$$

where $Y_1, Y_2, \ldots, Y_n$ are independent, Y_i and X_i have the same distribution for each i, and ϕ is a superadditive function such that the expectations in (1) exist. A sequence $\{X_n, n \geqslant 1\}$ of random variables is called a sequence of NSD random variables if for every $n \geqslant 1$, $(X_1, X_2, \ldots, X_n)$ is NSD. An array of random variables $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ is called an array of rowwise NSD random variables if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of NSD random variables.

Hu gave an example illustrating that NSD random variables are not necessarily NA and provided some basic properties and three structural theorems of NSD random variables[4]. Christofides and Vaggelatou[5] showed that NA random variables are NSD. Eghbal et al[6] derived two maximal inequalities and the strong law of large numbers for quadratic forms of NSD random variables under the assumption that $\{X_n, n \geqslant 1\}$ is a sequence of nonnegative NSD random variables with $EX_i^r < \infty$ for all $i \geqslant 1$ and some $r > 1$. Shen et al[7] discussed an almost sure convergence theorem and strong stability for weighted sums of NSD random variables.

A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$ Ref. 8. In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geqslant 1\}$ is independent. The sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite[8]. Erdős[9] proved the converse. These results are a fundamental theorem in probability theory which has been generalized and extended in several directions by many authors. One of the most important generalizations is the following theorem.

Theorem 1 (Baum and Katz[10]) *Let $\alpha > 1/2$ and $\alpha p > 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables. Assume further that $EX_1 = 0$ if $\alpha \leqslant 1$. Then $E|X_1|^p < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty.$$

for all $\varepsilon > 0$.

Motivated by Theorem 1, many authors studied the complete convergence for dependent random variables. See, for example, Refs. 11–18.

Definition 4 A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be *stochastically dominated* by a random variable X if there exists a positive constant C such that

$$\sup_{n\geqslant 1} P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$. An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of rowwise random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$\sup_{i\geqslant 1} P(|X_{ni}| > x) \leqslant CP(|X| > x) \tag{2}$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Definition 5 A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be *weakly mean dominated* by a random variable X if there exists a positive constant C such that

$$\frac{1}{n}\sum_{i=1}^{n} P(|X_i| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$. An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of rowwise random variables is said to be weakly mean dominated by a random variable X if there exists a positive constant C such that

$$\frac{1}{n}\sum_{i=1}^{n} P(|X_{ni}| > x) \leqslant CP(|X| > x) \tag{3}$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Obviously, the dominations of Definition 4 imply those of Definition 5, and Gut[12] gave an example which shows that the dominations of Definition 5 are strictly weaker than those of Definition 4.

The main purpose of the paper is to further study the complete convergence and complete moment convergence of weighted sums for an array of rowwise NSD random variables and a Baum-Katz-type theorem for sequences of NSD random variables. As an application, we obtain a Marcinkiewicz-Zygmund-type strong law of large numbers of weighted sums for NSD random variables. Our results extend and improve the corresponding ones of Ref. 11 and Ref. 17 and extend the corresponding ones of Ref. 12.

Throughout this paper, the symbols C, C_1, C_2, ... denote positive constants which may differ in various places. Let $I(A)$ be the indicator function of the set A. Let $x^+ = \max(0, x)$, $\operatorname{lm} x = \ln\max(x, 1)$, and $a_n = \mathcal{O}(b_n)$ stand for $|a_n| \leqslant C|b_n|$.

LEMMAS

To prove the main results of the paper, we need the following lemmas.

Lemma 1 (Wu[19]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $a > 0$ and $b > 0$*

$$E|X_n|^a I(|X_n| \leqslant b) \leqslant C_1\{E|X|^a I(|X| \leqslant b) + b^a P(|X| > b)\}$$

and

$$E|X_n|^a I(|X_n| > b) \leqslant C_2 E|X|^a I(|X| > b)$$

where C_1 and C_2 are positive constants.

Lemma 2 (Hu[4]) *If $(X_1, X_2, \dots, X_n)$ is NSD and $f_1, f_2, \dots, f_n$ are all nondecreasing, then $(f_1(X_1), f_2(X_2), \dots, f_n(X_n))$ is still NSD.*

Lemma 3 (Wang et al[17]) *Let $p > 1$ and $\{X_n, n \geqslant 1\}$ be a sequence of NSD random variables with $E|X_i|^p < \infty$ for each $i \geqslant 1$. Then for all $n \geqslant 1$, for $1 < p \leqslant 2$,*

$$E\left[\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|^p\right] \leqslant 2^{3-p}\sum_{i=1}^{n} E|X_i|^p$$

and for $p > 2$,

$$E\left[\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}X_i\right|^p\right]\leqslant 2\left(\frac{15p}{\ln p}\right)^p\left(\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right).$$

Lemma 4 (Sung[15]) *Let $\{Y_n, n\geqslant 1\}$ and $\{Z_n, n\geqslant 1\}$ be sequences of random variables. Then for any $q>1$, $\varepsilon>0$, and $a>0$,*

$$E\left[\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_i+Z_i)\right|-\varepsilon a\right]^+ \leqslant\left(\frac{1}{\varepsilon^q}+\frac{1}{q-1}\right)\frac{1}{a^{q-1}}E\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}Y_i\right|^q+E\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}Z_i\right|.$$

According to Lemma 1.10 of Wu[20], we get easily the following lemma.

Lemma 5 *Let $\{X_n, n\geqslant 1\}$ be a sequence of NSD random variables. Then there exists a positive constant C such that for any $x\geqslant 0$ and all $n\geqslant 1$*

$$\left[1-P\left(\max_{1\leqslant i\leqslant n}|X_i|>x\right)\right]^2\sum_{i=1}^{n}P(|X_i|>x)\leqslant CP\left(\max_{1\leqslant i\leqslant n}|X_i|>x\right).$$

MAIN RESULTS AND THEIR PROOFS

In this section, let $\psi(x)=1$ or $\psi(x)=\operatorname{lm} x$. Note that the function $\psi(x)$ has the following properties[21]: (a) for all $m\geqslant k\geqslant 1$,

$$\sum_{n=k}^{m}n^{r-1}\psi(n)\leqslant Cm^r\psi(m)\ \text{ if }\ r>0 \tag{4}$$

and

$$\sum_{n=m}^{\infty}n^{r-1}\psi(n)\leqslant Cm^r\psi(m)\ \text{ if }\ r<0; \tag{5}$$

(b) for all $p>0$,

$$\psi(|x|^p)\leqslant C(p)\psi(|x|)\leqslant C(p)\psi(1+|x|). \tag{6}$$

Theorem 2 *Let $\alpha>\frac{1}{2}$ and $\alpha p\geqslant 1$. Assume that $\{X_{ni}, i\geqslant 1, n\geqslant 1\}$ is an array of rowwise NSD random variables which is stochastically dominated by a random variable X. Assume that $\{a_{ni}, i\geqslant 1, n\geqslant 1\}$ is an array of real numbers with $\sum_{i=1}^{n}|a_{ni}|^q=\mathcal{O}(n)$ for some $q>\max\{(\alpha p-1)/(\alpha-1/2),2\}$. Assume further that $EX_{ni}=0$ for all $i\geqslant 1$ and $n\geqslant 1$ if $p\geqslant 1$. If*

$$E|X|^p\,\psi(|X|)<\infty, \tag{7}$$

then

$$\sum_{n=1}^{\infty}n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|T_{nj}\right|>\varepsilon n^{\alpha}\right)<\infty \tag{8}$$

for all $\varepsilon>0$, where $T_{nj}=\sum_{i=1}^{j}a_{ni}X_{ni}$.

Proof: Without loss of generality, we can assume that $a_{ni}>0$ for all $i\geqslant 1$ and $n\geqslant 1$. For fixed $n\geqslant 1$, let $X'_{ni}=-n^{\alpha}I(X_{ni}<-n^{\alpha})+X_{ni}I(|X_{ni}|\leqslant n^{\alpha})+n^{\alpha}I(X_{ni}>n^{\alpha})$ and $X''_{ni}=X_{ni}-X'_{ni}$, $i\geqslant 1$. Denote $S_{nj}=\sum_{i=1}^{j}a_{ni}(X'_{ni}-EX'_{ni})$, $H_{nj}=\sum_{i=1}^{j}a_{ni}(X''_{ni}-EX''_{ni})$ and $A_j=(j<|X|^{1/\alpha}\leqslant j+1)$, $j=0,1,2,\dots$. We will consider the following three cases.

(i) Let $p>1$. It is easy to check that

$$\begin{aligned}\sum_{n=1}^{\infty}&n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|T_{nj}\right|>\varepsilon n^{\alpha}\right)\\&\leqslant\sum_{n=1}^{\infty}n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|S_{nj}\right|>\frac{\varepsilon n^{\alpha}}{2}\right)\\&\quad+\sum_{n=1}^{\infty}n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|H_{nj}\right|>\frac{\varepsilon n^{\alpha}}{2}\right)\\&:=I^*+J^*.\end{aligned}$$

In view of the C_r inequality and $\sum_{i=1}^{n}a_{ni}^q=\mathcal{O}(n)$, we have that for all $0<\gamma\leqslant q$,

$$\frac{1}{n}\sum_{i=1}^{n}a_{ni}^{\gamma}\leqslant\left(\frac{1}{n}\sum_{i=1}^{n}a_{ni}^{q}\right)^{\gamma/q}=\mathcal{O}(1). \tag{9}$$

For J^*, noting that $\left|X''_{ni}\right|\leqslant|X_{ni}|I(|X_{ni}|>n^{\alpha})$, we have by Markov's inequality, Lemma 1, and (6) that

$$\begin{aligned}J^*&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}\psi(n)\sum_{i=1}^{n}a_{ni}E\left|X''_{ni}\right|\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X|>n^{\alpha})\\&=C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha}\psi(n)\sum_{j=n}^{\infty}E|X|I(A_j)\\&=C\sum_{j=1}^{\infty}E|X|I(A_j)\sum_{n=1}^{j}n^{\alpha p-1-\alpha}\psi(n)\end{aligned}$$

$$\leqslant C\sum_{j=1}^{\infty} j^{\alpha p-\alpha}\psi(j)E|X|I(A_j)$$
$$\leqslant CE|X|^p\psi(|X|)<\infty. \tag{10}$$

For I^*, $\{a_{ni}X'_{ni}-Ea_{ni}X'_{ni}, i\geqslant 1\}$ is a sequence of NSD random variables for each $n\geqslant 1$ following from Lemma 2. Based on Markov's inequality, Lemma 3, and Jensen's inequality, we have that for any $r\geqslant 2$,

$$I^*\leqslant C_r\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)E\left(\max_{1\leqslant j\leqslant n}\left|S_{nj}\right|^r\right)$$
$$\leqslant C_r\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)\sum_{i=1}^{n} a_{ni}^r E\left|X'_{ni}\right|^r$$
$$+C_r\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)\left[\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni})^2\right]^{r/2}$$
$$:= I_1^*+I_2^*. \tag{11}$$

Case 1: $\alpha>\frac{1}{2}$, $\alpha p>1$ and $p\geqslant 2$. Take $r=q$. By $q>\max\{(\alpha p-1)/(\alpha-1/2),2\}$, it follows that $q>p$ and $\alpha p-2-\alpha q+\frac{1}{2}q<-1$. For I_1^*, we have by the C_r inequality that

$$I_1^*\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q}\psi(n)E|X|^q I(|X|\leqslant n^\alpha)$$
$$+C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X|>n^\alpha)$$
$$\leqslant C\sum_{j=1}^{\infty} j^{\alpha q}P(A_{j-1})\sum_{n=j}^{\infty} n^{\alpha(p-q)-1}\psi(n)+C$$
$$\leqslant C\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(A_{j-1})+C$$
$$\leqslant CE|X|^p\psi(|X|)+C<\infty. \tag{12}$$

For I_2^*, note that $EX^2<\infty$ from $E|X|^p\psi(|X|)<\infty$ for $p\geqslant 2$. We have by (9) that

$$I_2^*\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2\right)^{q/2}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^2 EX^2\right)^{q/2}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2}\psi(n)<\infty.$$

Case 2: $\alpha>1/2$, $\alpha p>1$ and $1<p<2$. Take $r=2$. As with the proofs of (10)–(12), we have that

$$I^*\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}\psi(n)EX^2 I(|X|\leqslant n^\alpha)$$
$$+C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X|>n^\alpha)$$
$$<\infty. \tag{13}$$

Case 3: $\alpha>\frac{1}{2}$, $\alpha p=1$ and $p>1$. Take $r=2$. Note that $\frac{1}{2}<\alpha<1$ if $\alpha p=1$. As with the proof of (13), it follows that $I^*<\infty$.

(ii) Let $p=1$. Note that $\alpha\geqslant 1$ from $\alpha p\geqslant 1$. By $EX_{ni}=0$ for $i\geqslant 1$ and $n\geqslant 1$, Lemma 1, (9), and (7) we have that

$$n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}EX'_{ni}\right|$$
$$\leqslant n^{-\alpha}\sum_{i=1}^{n} a_{ni}E|X_{ni}|I(|X_{ni}|>n^\alpha)$$
$$\leqslant n^{1-\alpha}E|X|I(|X|>n^\alpha)\to 0 \text{ as } n\to\infty.$$

Hence for n large enough, we have

$$n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}EX'_{ni}\right|<\frac{\varepsilon}{2}.$$

It follows that

$$\sum_{n=1}^{\infty} n^{\alpha-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|T_{nj}\right|>\varepsilon n^\alpha\right)$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha-1}\psi(n)P(|X|>n^\alpha)$$
$$+C\sum_{n=1}^{\infty} n^{\alpha-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|S_{nj}\right|>\frac{\varepsilon n^\alpha}{2}\right)$$
$$:= CI_1+CI_2. \tag{14}$$

For I_1, we have by (4) and (7) that

$$I_1=\sum_{n=1}^{\infty} n^{\alpha-1}\psi(n)\sum_{i=n}^{\infty} P(A_i)$$
$$=\sum_{i=1}^{\infty} P(A_i)\sum_{n=1}^{i} n^{\alpha-1}\psi(n)$$
$$\leqslant C\sum_{i=1}^{\infty} P(A_i)i^\alpha\psi(i)$$
$$\leqslant CE|X|\psi(|X|)<\infty. \tag{15}$$

For I_2, we have by Markov's inequality, Lemma 3 and Lemma 1, (5) and (6) that

$$\begin{aligned} I_2 &\leqslant C\sum_{n=1}^{\infty} n^{-\alpha-2}\psi(n)E\max_{1\leqslant j\leqslant n} S_{nj}^2 \\ &\leqslant C\sum_{n=1}^{\infty} n^{-\alpha-2}\psi(n)\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni})^2 \\ &\leqslant C\sum_{n=1}^{\infty} n^{-\alpha-1}\psi(n)EX^2 I(|X|\leqslant n^{\alpha}) \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha-1}\psi(n)P(|X|>n^{\alpha}) \\ &= C\sum_{k=1}^{\infty} EX^2 I(A_{k-1})\sum_{n=k}^{\infty} n^{-\alpha-1}\psi(n)+C \\ &\leqslant C\sum_{k=1}^{\infty} k^{-\alpha}\psi(k)EX^2 I(A_{k-1})+C \\ &\leqslant CE|X|\psi(|X|)+C<\infty. \end{aligned} \tag{16}$$

By (14)–(16), (8) holds for the case $p=1$.

(iii) Let $0<p<1$. Denote

$$\begin{aligned} T_{nj} &= \sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}|\leqslant n^{\alpha}) \\ &\quad + \sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}|>n^{\alpha}) \\ &=: S'_{nj}+S''_{nj}. \end{aligned} \tag{17}$$

Noting that $E|X|^p\psi(|X|)<\infty$, we have by Markov's inequality, Lemma 1, (5)–(7) and (9) that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|S'_{nj}\right|>\varepsilon n^{\alpha}\right) \\ &\leqslant \frac{C}{\varepsilon}\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X|\leqslant n^{\alpha}) \\ &\quad + \frac{C}{\varepsilon}\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n)P(|X|>n^{\alpha}) \\ &\leqslant \frac{C}{\varepsilon}\sum_{j=1}^{\infty} j^{\alpha}P(A_{j-1})\sum_{n=j}^{\infty} n^{\alpha p-1-\alpha}\psi(n) \\ &\quad + \frac{C}{\varepsilon}\sum_{j=1}^{\infty} P(A_j)\sum_{n=1}^{j} n^{\alpha p-1}\psi(n) \\ &\leqslant \frac{C}{\varepsilon}\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(A_{j-1}) \\ &\quad + \frac{C}{\varepsilon}\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(A_j) \\ &\leqslant CE|X|^p\psi(|X|)<\infty \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|S''_{nj}\right|>\varepsilon n^{\alpha}\right) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p/2-1}\psi(n)E|X|^{p/2}I(|X|>n^{\alpha}) \\ &= C\sum_{n=1}^{\infty} n^{\alpha p/2-1}\psi(n)\sum_{j=n}^{\infty} E|X|^{p/2}I(A_j) \\ &\leqslant C\sum_{j=1}^{\infty} j^{\alpha p/2}P(A_j)\sum_{n=1}^{j} n^{\alpha p/2-1}\psi(n) \\ &\leqslant C\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(A_{j-1}) \\ &\leqslant CE|X|^p\psi(|X|)<\infty. \end{aligned} \tag{19}$$

Hence (17)–(19) implies (8). From all the statements above, we have proved (8).

□

Remark 1 Under the conditions of Theorem 2, we have that for $p>1$

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\left(\max_{1\leqslant j\leqslant n}\left|T_{nj}\right|-\varepsilon n^{\alpha}\right)^{+}<\infty.$$

In fact, by Lemma 4 with $r\geqslant 2$

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\left(\max_{1\leqslant j\leqslant n}\left|T_{nj}\right|-\varepsilon n^{\alpha}\right)^{+} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)E\left(\max_{1\leqslant j\leqslant n}\left|S_{nj}\right|\right)^{r} \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\left(\max_{1\leqslant j\leqslant n}\left|H_{nj}\right|\right). \end{aligned}$$

The rest is the same as the method of the proof of Theorem 2 in the case $p>1$.

Remark 2 Taking $\psi(x)\equiv 1$ and $a_{ni}\equiv 1$ in Theorem 2, we can get (i) of Corollary 3.2 of Ref. 17. Meanwhile, the case $\frac{1}{2}<\alpha\leqslant 1$, $p>1$ and $\alpha p>1$ is extended to the case $\alpha>\frac{1}{2}$, $\alpha p\geqslant 1$. Taking $\psi(x)\equiv 1$, $a_{ni}\equiv 1$ and $\alpha=1$, $p=1$ in Theorem 2, we can get (ii) of Corollary 3.2 of Ref. 17 and weaken the condition $E|X|\ln|X|<\infty$ to the condition $E|X|<\infty$. Hence we extend and improve the corresponding results of Ref. 17. This is stated by the case of the following corollary.

Corollary 1 *Let $\alpha \geqslant \frac{1}{2}$ and $\alpha p \geqslant 1$. Let $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of rowwise NSD random variables which is stochastically dominated by a random variable X and $EX_{ni} = 0$ for all $i \geqslant 1$, $n \geqslant 1$ if $p \geqslant 1$. If $E|X|^p < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon n^{\alpha}\right) < \infty.$$

Corollary 2 *If the condition (2) of Corollary 1 is replaced by the strictly weaker condition (3), then Corollary 1 still holds.*

Taking $\alpha = 1/t$ and $p = 2t$ for $0 < t < 2$ and $\psi(x) = 1$ in Theorem 2, we can get the following corollary.

Corollary 3 *Assume that $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of rowwise NSD random variables which is stochastically dominated by a random variable X. Assume that $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of real numbers with $\sum_{i=1}^{n} |a_{ni}|^q = \mathcal{O}(n)$ for some $q > \max\{(\alpha p - 1)/(\alpha - 1/2), 2\}$. Assume further that $EX_{ni} = 0$ for all $1 \leqslant i \leqslant n$ and $n \geqslant 1$. If $E|X|^{2t} < \infty$ for $0 < t < 2$, then*

$$\sum_{n=1}^{\infty} P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > \varepsilon n^{1/t}\right) < \infty$$

for all $\varepsilon > 0$.

Remark 3 Taking $a_{ni} \equiv 1$, we get the result of Ref. 11 for the case of a sequence of NSD random variables.

As with the proof of Theorem 2, we can get easily the following result.

Theorem 3 *Let $\alpha > \frac{1}{2}$ and $\alpha p \geqslant 1$. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of NSD random variables which is stochastically dominated by a random variable X. Assume that $\{a_n, n \geqslant 1\}$ is a sequence of real numbers with $\sum_{i=1}^{n} |a_i|^q = \mathcal{O}(n)$ for some $q > \max\{(\alpha p - 1)/(\alpha - 1/2), 2\}$. If (7) holds then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \psi(n) P\left(\max_{1\leqslant j\leqslant n} \left|S_j\right| > \varepsilon n^{\alpha}\right) < \infty \qquad (20)$$

for all $\varepsilon > 0$, where $S_j = \sum_{i=1}^{j} a_i X_i$.

Remark 4 Theorems 2 and 3 are obtained for $\psi(x) = 1$ or $\psi(x) = \operatorname{lm} x$. In fact, if $\psi(x)$ is a slowing varying function at infinite, according to Lemma 1 of Bai and Su[22], Theorems 2 and 3 can be also obtained.

In the following, we give the Marcinkiewicz-Zygmund type strong law of large numbers of weights sums for sequences of NSD random variables.

Corollary 4 *Let $\alpha > \frac{1}{2}$ and $\alpha p \geqslant 1$. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of NSD random variables which is stochastically dominated by a random variable X. Assume that $\{a_n, n \geqslant 1\}$ is a sequence of real numbers with $\sum_{i=1}^{n} |a_i|^q = \mathcal{O}(n)$ for some $q > \max\{(\alpha p-1)/(\alpha-1/2), 2\}$. If $E|X|^p < \infty$, then for all $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n} \left|S_j\right| > \varepsilon n^{\alpha}\right) < \infty \qquad (21)$$

and

$$n^{-\alpha} S_n \to 0 \;\; a.s. \;\; n \to \infty. \qquad (22)$$

Further, for $p > 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leqslant j\leqslant n} \left|S_j\right| - \varepsilon n^{\alpha}\right)^{+} < \infty \qquad (23)$$

for all $\varepsilon > 0$.

Proof: Taking $\psi(x) = 1$ in Theorem 3, we can get (21) easily. As with the proof of Remark 1, (23) is obtained immediately. We only need to prove (22).

By (22), it follows that for all $\varepsilon > 0$,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n} \left|S_j\right| > \varepsilon n^{\alpha}\right) \\
&= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n} \left|S_j\right| > \varepsilon n^{\alpha}\right) \\
&\geqslant \begin{cases} \sum_{k=0}^{\infty} (2^k)^{\alpha p-2} 2^k P(B_k), & \text{if } \alpha p \geqslant 2, \\ \sum_{k=0}^{\infty} (2^{k+1})^{\alpha p-2} 2^k P(B_k), & \text{if } 1 \leqslant \alpha p < 2 \end{cases} \\
&\geqslant \begin{cases} \sum_{k=0}^{\infty} P(B_k), \text{if } \alpha p \geqslant 2, \\ \frac{1}{2} \sum_{k=0}^{\infty} P(B_k), \text{if } 1 \leqslant \alpha p < 2, \end{cases}
\end{aligned}$$

where $B_k = (\max_{1\leqslant j\leqslant 2^k} \left|S_j\right| > \varepsilon 2^{(k+1)\alpha})$.

According to the Borel-Cantelli lemma, we obtain

$$\frac{\max_{1\leqslant j\leqslant 2^k} \left|S_j\right|}{2^{(k+1)\alpha}} \to 0 \;\; a.s. \;\; k \to \infty. \qquad (24)$$

For all positive integers n, there exists a positive integer k such that $2^{k-1} \leqslant n \leqslant 2^k$. We have by (24) that

$$\begin{aligned}
n^{-\alpha} |S_n| &\leqslant \max_{2^{k-1}\leqslant n\leqslant 2^k} n^{-\alpha} |S_n| \\
&\leqslant \frac{2^{\alpha} \max_{1\leqslant j\leqslant 2^k} \left|S_j\right|}{2^{(k+1)\alpha}} \to 0 \;\; a.s. \;\; k \to \infty,
\end{aligned}$$

which implies that

$$n^{-\alpha}S_n \to 0 \ \ a.s. \ \ n \to \infty.$$

□

Remark 5 Taking $a_n \equiv 1$ for $n \geqslant 1$ in Corollary 3, we can get a Baum-Katz type result for sequences of NSD random variables. In addition, the case $\alpha p = 1$ and the case $\alpha p \geqslant 1$ and $0 < p \leqslant 1$ are also considered. Taking $\alpha = 1$ and $p = 2$ in Corollary 3, we can get a Hsu-Robbins type theorem[8] for NSD random variables.

Taking $a_i \equiv 1$, and $\alpha = 1$ and $p = 1$ in Corollary 3, we get the following corollary immediately.

Corollary 5 *Let $\{X_n, n \geqslant 1\}$ be a sequence of NSD random variables which is stochastically dominated by a random variable X and $EX_n = 0$ for all $n \geqslant 1$. If $E|X| < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n\right) < \infty$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0 \text{ a.s. as } n \to \infty.$$

Theorem 4 *Let $\alpha > \frac{1}{2}$ and $\alpha p \geqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of NSD random variables. Assume that $\{a_n, n \geqslant 1\}$ is a sequence of real numbers with $\sum_{i=1}^{n} |a_i|^q = \mathcal{O}(n)$ for some $q > \max\{(\alpha p - 1)/(\alpha - 1/2), 2\}$. Assume further that $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. If there exist a random variable X and positive numbers C_1 and C_2 such that for all $x \geqslant 0$, $n \geqslant 1$,*

$$\begin{aligned} C_1 P(|X| \geqslant x) &\leqslant \inf_{i\geqslant 1} P(|X_i| \geqslant x) \\ &\leqslant \sup_{i\geqslant 1} P(|X_i| \geqslant x) \leqslant C_2 P(|X| \geqslant x), \end{aligned} \quad (25)$$

then (7) is equivalent to (20).

Proof: By Theorem 3, we can see that (7) implies (20) under the conditions of Theorem 4. So we only need to prove that (20) implies (7). By (20), taking $a_i \equiv 1$ for all $i \geqslant 1$, it follows that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n) P\left(\max_{1\leqslant j\leqslant n}\left|X_j\right| > \varepsilon n^{\alpha}\right) < \infty. \quad (26)$$

Note that for $n \geqslant 3$,

$$\begin{aligned} &P\left(\max_{1\leqslant j\leqslant n}\left|X_j\right| > \varepsilon n^{\alpha}\right) \\ &\leqslant n^{\alpha p-1}\psi(n) P\left(\max_{1\leqslant j\leqslant n}\left|X_j\right| > \varepsilon n^{\alpha}\right) \\ &\leqslant C\sum_{i=n}^{2n} i^{\alpha p-2}\psi(i) P\left(\max_{1\leqslant j\leqslant i}\left|X_j\right| > \left(\frac{\varepsilon}{2^{\alpha}}\right) i^{\alpha}\right). \end{aligned}$$

We have by (26) that

$$P\left(\max_{1\leqslant j\leqslant n}\left|X_j\right| > \varepsilon n^{\alpha}\right) \to 0 \ \text{ as } \ n \to \infty. \quad (27)$$

Using Lemma 5, we have

$$\begin{aligned} &\left[1 - P\left(\max_{1\leqslant i\leqslant n}|X_i| > \varepsilon n^{\alpha}\right)\right]^2 \sum_{i=1}^{n} P(|X_i| > \varepsilon n^{\alpha}) \\ &\qquad \leqslant CP\left(\max_{1\leqslant i\leqslant n}|X_i| > \varepsilon n^{\alpha}\right). \end{aligned} \quad (28)$$

Combining (25) with (27) and (28), we have that for all $\varepsilon > 0$

$$\begin{aligned} nP(|X| > \varepsilon n^{\alpha}) &\leqslant C\sum_{i=1}^{n} P(|X_i| > \varepsilon n^{\alpha}) \\ &\leqslant CP\left(\max_{1\leqslant i\leqslant n}|X_i| > \varepsilon n^{\alpha}\right). \end{aligned} \quad (29)$$

Take $\varepsilon = 1$. It follows that from (26), (29) and (4)

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n) P\left(\max_{1\leqslant i\leqslant n}|X_i| > n^{\alpha}\right) \\ &\geqslant C\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n) P(|X| > n^{\alpha}) \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n)\sum_{j=n}^{\infty} P(A_j) \\ &= C\sum_{j=1}^{\infty} P(A_j)\sum_{n=1}^{j} n^{\alpha p-1}\psi(n) \\ &\geqslant C\sum_{j=1}^{\infty} P(A_j) j^{\alpha p}\psi(j) \\ &\geqslant CE|X|^p \psi(|X|), \end{aligned}$$

where $A_j = (j < |X|^{1/\alpha} \leqslant j+1)$, $j \geqslant 0$, i.e., (7) holds. □

If the condition (2) of Theorem 2 is replaced by the weaker condition (3), we will get the result of an array of NSD random variables, which is proved in a similar way to the proof of Theorem 2.

Theorem 5 *Let $\alpha > \frac{1}{2}$ and $\alpha p \geqslant 1$. Assume that $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of rowwise NSD random variables which is weakly mean dominated by a random variable X. Assume further that $EX_{ni} = 0$ for all $i \geqslant 1$ and $n \geqslant 1$ if $p \geqslant 1$. If (7) holds, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon n^{\alpha}\right) < \infty$$

for all $\varepsilon > 0$.

Taking $\alpha = 1/t$ and $p = 2t$ for $0 < t < 2$ in Theorem 4, the following corollary is obtained immediately.

Corollary 6 *Assume that $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of rowwise NSD random variables which is weakly mean dominated by a random variable X. Assume further that $EX_{ni} = 0$ for all $1 \leqslant i \leqslant n$ and $n \geqslant 1$. If $E|X|^{2t} < \infty$ for $0 < t < 2$, then*

$$\sum_{n=1}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon n^{1/t}\right) < \infty$$

for all $\varepsilon > 0$.

Remark 6 Corollary 5 extends Theorem 2.1 of Ref. 12 for independent random variables to the case of NSD random variables and extends and improves Theorem 2 of Ref. 11.

Theorem 6 *Let $\alpha > \frac{1}{2}$ and $\alpha p \geqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of NSD random variables. Assume further that $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. If there exist a random variable X and positive numbers C_1 and C_2 such that for all $x \geqslant 0$, $n \geqslant 1$,*

$$C_1 P(|X| \geqslant x) \leqslant \frac{1}{n}\sum_{i=1}^{n} P(|X_i| \geqslant x) \leqslant C_2 P(|X| \geqslant x).$$

Then (7) is equivalent to (20).

Corollary 7 *Let $\alpha > \frac{1}{2}$ and $\alpha p \geqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed NSD random variables. Assume that $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. Then $E|X_1|^p < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty$$

for all $\varepsilon > 0$.

Remark 7 Corollary 6 extends the Baum-Katz theorem (i.e., Theorem 1) for i.i.d. random variables to the case of NSD random variables. In addition, it is the complement to the case $\alpha p = 1$ and $\alpha > \frac{1}{2}$.

Corollary 8 *Let $\alpha > \frac{1}{2}$, $\alpha p \geqslant 1$ and $p > 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of zero mean and identically distributed NSD random variables. Then $E|X_1|^p < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty \quad (30)$$

for all $\varepsilon > 0$.

Proof: Taking $\psi(x) = 1$ in Theorem 2, if $E|X_1|^p < \infty$, then (30) follows from Theorem 2 and Remark 1 immediately. We only need to prove that (30) implies $E|X_1|^p < \infty$.

It is easy to check that

$$\varepsilon n^{\alpha} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| > 2\varepsilon n^{\alpha}\right) \leqslant E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha}\right)^{+}.$$

Hence (30) implies that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty$$

for all $\varepsilon > 0$. The rest of the proof is similar to that of Theorem 4 and is omitted. □

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