RESEARCH ARTICLE
ScienceAsia 39 (2013): **194–203**
doi: 10.2306/scienceasia1513-1874.2013.39.194

Large-time behaviour for the compressible Navier-Stokes equations with a non-autonomous external force and a heat source

Baowei Feng[a], **Xin-guang Yang**[b,*]

[a] College of Information Science and Technology, Donghua University, Shanghai 201620, China
[b] College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, China

*Corresponding author, e-mail: yangxinguangyxg@yahoo.cn

Received 28 Nov 2012
Accepted 18 Jan 2013

ABSTRACT: In this paper, we study the global existence of solutions for the compressible Navier-Stokes equations with a non-autonomous external force and a heat source in H^4. Under suitable assumptions, we obtain the large-time behaviour of solutions in H^4.

KEYWORDS: global existence, uniform estimate, asymptotic behaviour

INTRODUCTION

In this paper, we are concerned with the global existence and large-time behaviour of solutions to the following 1-d compressible Navier-Stokes equations with a non-autonomous external force and a heat source in Lagrangian coordinates:

$$u_t = v_x, \tag{1}$$

$$v_t = \sigma_x + f\left(\int_0^x u\,\mathrm{d}y, t\right), \tag{2}$$

$$e_t = \sigma v_x - q_x + g\left(\int_0^x u\,\mathrm{d}y, t\right), \tag{3}$$

where $x \in [0,1]$, u denotes the specific volume (i.e., $u = 1/\rho$), v the velocity, θ the absolute temperature, σ the stress, e the internal energy, and q the heat flux. The functions f, g are non-autonomous external force and the heat source.

In this paper, we only investigate the polytropic viscous ideal gas, i.e.,

$$e = c_v\theta,\ \sigma = -p + \mu\frac{v_x}{u},\ q = -\kappa\frac{\theta_x}{u},\ p = -R\frac{\theta}{u}, \tag{4}$$

where the coefficients c_v, μ, κ, R are positive constants.

We consider a typical initial boundary value problem for (1)–(4) in the reference domain $Q := \Omega \times [0,+\infty) = [0,1] \times [0,+\infty)$ under the Dirichlet-Neumann boundary conditions for the fluid unknowns

$$v(0,t) = v(1,t) = 0,\ q(0,t) = q(1,t) = 0,\ t \geqslant 0, \tag{5}$$

and initial conditions

$$t = 0:\ u = u_0(x),\ v = v_0(x),\ \theta = \theta_0(x). \tag{6}$$

In recent years, many mathematicans have paid attention to the Navier-Stokes equations. Firstly, we recall some previous work concerning the related results. When the temperature θ is a constant, i.e., the system only contains (1)–(2), for $f = 0$ and $g = 0$, Kanel[1], Itaya[2], Kazhikhov[3], Kazhikhov and Nikolaev[4,5], Kazhikhov and Shelukhin[6], etc., have studied the global existence and uniqueness of the uniformly boundary, global-in-time solution under various initial conditions and the equation of state and so on. For $f \neq 0$ and $g = 0$, Mucha[7] considered the compressible barotropic Navier-Stokes system in monodimensional case with a Neumann boundary condition given on a free boundary and proved the global existence with uniform boundedness for large initial data and a positive force. Moreover, when $t \to \infty$, the author obtained the solutions tended to the stationary solution. Zhang and Fang[8] studied a free boundary problem for compressible Navier-Stokes equations with density-dependent viscosity. Under certain assumptions imposed on the initial data, the authors obtained the global existence and uniqueness of the weak solution and showed that it converged to a stationary one as time tends to infinity. Later on, Qin and Zhao[9] obtained the global existence and asymptotic behaviour of solutions in $H^i (i = 1, 2)$ to an initial boundary value problem in a bounded region. When the temperature θ is not a constant, Qin et al[10] proved the regularity and continuous dependence on initial data in $H^i (i = 1,2,4)$ for large initial data and then showed the large-time behaviour of solutions in $H^i (i = 2,4)$ for small initial data to the Cauchy problem. Zheng and Qin[11] obtained the existence of maximal attractor for the problem for $f \equiv 0,\ g \equiv 0$. For more results, we can refer to Refs. 12–14.

For system (1)–(6), Qin and Yu[15] proved the global existence and large-time behaviour of solutions in $H^i (i = 1,2)$ in a bounded region. But the global existence and large-time behaviour of solutions in H^4 are still open. So in this paper, we study the global existence and large-time behaviour of solutions in H^4.

The aim of this paper is to establish the global existence and large-time behaviour of solutions to the system (1)–(6). We shall firstly establish the global existence in H^4, and then we shall prove large-time behaviour of solutions in H^4.

In this paper, we assume for any $x \in \Omega$

$$\int_0^1 u_0(x)\,\mathrm{d}x := \overline{u_0}, \quad 0 < C_0^{-1} \leqslant u_0(x) \leqslant C_0, \tag{7}$$

where C_0 is a positive constant. Moreover, we suppose that for any $u(x,\cdot) \in L^\infty(\mathbb{R}^+, L^1(\Omega))$ with $\xi(x,t) = \int_0^x u(y,t)\,\mathrm{d}y$ and $\widehat{f}(x,t) = \int_0^t f(\int_0^x u(y,s)\,\mathrm{d}y, s)\,\mathrm{d}s$, the non-autonomous external force $f = f(\xi(x,t),t)$ and heat source $g = g(\xi(x,t),t)$ satisfy the following conditions:

$$\begin{aligned} f \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \cap L^2(\mathbb{R}^+, L^\infty(\Omega)) \\ \cap L^1(\mathbb{R}^+, L^1(\Omega)), \end{aligned} \tag{8}$$

$$\widehat{f} \in L^1(\mathbb{R}^+, L^2(\Omega)),$$

$$f_\xi \in L^2(\mathbb{R}^+, L^2(\Omega)) \cap L^\infty(\mathbb{R}^+, L^2(\Omega)), \tag{9}$$

$$f_t,\ f_{\xi\xi} \in L^2(\mathbb{R}^+, L^2(\Omega)) \cap L^\infty(\mathbb{R}^+, L^2(\Omega)), \tag{10}$$

$$f_{\xi t},\ f_{tt},\ f_{\xi\xi\xi} \in L^2(\mathbb{R}^+, L^2(\Omega)), \tag{11}$$

$$\begin{aligned} g > 0,\ g \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \cap L^2(R^+, L^2(\Omega)) \\ \cap L^1(\mathbb{R}^+, L^\infty(\Omega)), \end{aligned} \tag{12}$$

$$\begin{aligned} g_\xi,\ g_t,\ g_{\xi\xi} \in L^2(R^+, L^2(\Omega)) \\ \cap L^\infty(\mathbb{R}^+, L^2(\Omega)), \end{aligned} \tag{13}$$

$$g_{\xi t},\ g_{tt},\ g_{\xi\xi\xi} \in L^2(\mathbb{R}^+, L^2(\Omega)). \tag{14}$$

The notation in this paper will be as follows. L^q, $1 \leqslant q \leqslant +\infty$, $W^{m,q}$, $m \in N$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on Ω. In addition, $\|\cdot\|_B$ denotes the norm in the space B; we also put $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Subscripts t and x denote the (partial) derivatives with respect to t and x, respectively. We use C_i, $(i = 1,2,3,4)$ to denote the generic positive constants depending on the $\|(u_0, v_0, \theta_0)\|_{H^i\times H^i\times H^i}$, $\min_{x\in[0,1]} v_0(x)$, $\min_{x\in[0,1]} \theta_0(x)$, but not depending on t. We denote $v_{3x} := v_{xxx}$ and $v_{3xt} := v_{xxxt}$.

Our result in this paper reads as follows.

Theorem 1 *Let (8)–(14) hold. Suppose that* $(u_0, v_0, \theta_0) \in H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$ *with* $u_0 > 0$ *and* $\theta_0 > 0$ *for any* $x \in [0,1]$, *and that the compatibility conditions hold. Then there exists a unique global solution* $(u(t), v(t), \theta(t)) \in L^\infty([0,+\infty), H^4(0,1) \times H_0^4(0,1) \times H^4(0,1))$ *to the problem (1)–(6) verifying that for any* $t > 0$,

$$\begin{aligned} &\|u(t) - \overline{u}\|_{H^4}^2 + \|u_t(t)\|_{H^3}^2 + \|u_{tt}(t)\|_{H^1}^2 \\ &+ \|v(t)\|_{H^4}^2 + \|v_{tt}(t)\|^2 + \|\theta(t) - \overline{\theta}\|_{H^4}^2 \\ &+ \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}(t)\|^2 + \int_0^t (\|u - \overline{u}\|_{H^4}^2 + \|v\|_{H^5}^2 \\ &+ \|v_t\|_{H^3}^2 + \|v_{tt}\|_{H^1}^2 + \|\theta - \overline{\theta}\|_{H^5}^2 + \|\theta_t\|_{H^3}^3 \\ &+ \|\theta_{tt}\|_{H^1}^2)(s)\,\mathrm{d}s \leqslant C_4, \end{aligned} \tag{15}$$

$$\int_0^t (\|u_t\|_{H^4}^2 + \|u_{tt}\|_{H^2}^2 + \|u_{3t}\|^2)(s)\,\mathrm{d}s \leqslant C_4. \tag{16}$$

Moreover, as $t \to +\infty$, *we have*

$$\|u(t) - \overline{u}\|_{H^4} \to 0, \quad \|v(t)\|_{H^4} \to 0, \quad \|\theta(t) - \overline{\theta}\|_{H^4} \to 0, \tag{17}$$

where $\overline{u} = \int_0^1 u(x,t)\,\mathrm{d}x = \int_0^1 u_0\,\mathrm{d}x$, $\overline{\theta} = \int_0^1 \theta(x,t)\,\mathrm{d}x$.

Corollary 1 *The global solution* $(u(t), v(t), \theta(t))$ *obtained in Theorem 1 is in fact a classical solution and as* $t \to +\infty$, *we have*

$$\|(u(t) - \overline{u}, v(t), \theta(t) - \overline{\theta})\|_{(C^{3+1/2})^3} \to 0.$$

Remark 1 Theorem 1 also holds for the boundary conditions

$$v(0,t) = v(1,t) = 0, \quad \theta(0,t) = \theta(1,t) = T_0 > 0$$

where $\overline{\theta} > 0$ can be replaced by $T_0 = const$.

GLOBAL EXISTENCE IN $H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$

In this section, we shall establish the global existence in $H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$.

First we give the global existence of solutions in $H^1(0,1) \times H_0^1(0,1) \times H^1(0,1)$ and $H^2(0,1) \times H_0^2(0,1) \times H^2(0,1)$ established in Ref. 15.

Lemma 1 *Under the assumptions* (8)–(14)*, for any* $(u_0, v_0, \theta_0) \in H^1(0,1) \times H_0^1(0,1) \times H^1(0,1)$ *with* $u_0 > 0$ *and* $\theta_0 > 0$ *for any* $x \in [0,1]$*, and that the compatibility conditions hold. Then the problem* (1)–(6) *admits a unique global solution* $(u(t), v(t), \theta(t)) \in H^1(0,1) \times H_0^1(0,1) \times H^1(0,1)$ *such that*

$$0 < C_1^{-1} \leqslant u(x,t) \leqslant C_1, \tag{18}$$

$$0 < C_1^{-1} \leqslant \theta(x,t) \leqslant C_1,\ \forall (x,t) \in Q,$$

$$\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + \int_0^t (\|u_x\|^2 + \|v_x\|^2 + \|v_t\|^2 + \|v_{xx}\|^2 + \|\theta_x\|^2 + \|\theta_t\|^2 + \|\theta_{xx}\|^2)(s)\,\mathrm{d}s \leqslant C_1,\ \ \forall\, t > 0. \tag{19}$$

Lemma 2 *Under the assumptions* (8)–(14)*, for any* $(u_0, v_0, \theta_0) \in H^2(0,1) \times H_0^2(0,1) \times H^2(0,1)$ *with* $u_0 > 0$ *and* $\theta_0 > 0$ *for any* $x \in [0,1]$*, and that the compatibility conditions hold. Then the problem* (1)–(6) *admits a unique global solution* $(u(t), v(t), \theta(t)) \in H^2(0,1) \times H_0^2(0,1) \times H^2(0,1)$ *such that*

$$\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \int_0^t (\|u_x\|_{H^1}^2 + \|v_x\|_{H^2}^2 + \|v_t\|_{H^1}^2 + \|\theta_x\|_{H^2}^2 + \|\theta_t\|_{H^1}^2)(s)\,\mathrm{d}s \leqslant C_1,\ \ \forall\, t > 0. \tag{20}$$

Now we are ready to establish the global existence in $H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$.

Lemma 3 *Under the assumptions in Theorem 1, for any* $(u_0, v_0, \theta_0) \in H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$*, the following estimates hold*

$$\|v_{xt}(x,0)\| + \|\theta_{xt}(x,0)\| \leqslant C_3, \tag{21}$$

$$\|v_{tt}(x,0)\| + \|\theta_{tt}(x,0)\| + \|v_{xxt}(x,0)\| + \|\theta_{xxt}(x,0)\| \leqslant C_4, \tag{22}$$

$$\|v_{tt}(t)\|^2 + \int_0^t \|v_{xtt}(s)\|^2\,\mathrm{d}s \leqslant C_4 + C_4 \int_0^t \|\theta_{xxt}(s)\|^2\,\mathrm{d}s, \tag{23}$$

$$\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{xtt}(s)\|^2\,\mathrm{d}s \leqslant C_4\varepsilon^{-3} + C_2\varepsilon^{-1} \int_0^t \|\theta_{xxt}(s)\|^2\,\mathrm{d}s + C_1\varepsilon \int_0^t (\|v_{xtt}\|^2 + \|v_{xxt}\|^2)(s)\,\mathrm{d}s. \tag{24}$$

Proof: By Lemma 1 and Lemma 2, we derive from (2)

$$\begin{aligned}\|v_t(t)\| &\leqslant C_1(\|u_x(t)\| + \|\theta_x(t)\| + \|v_{xx}(t)\| + \|v_x(t)\|_{L^\infty}\|u_x(t)\| + \|f(t)\|)\\ &\leqslant C_2(\|v_x(t)\|_{H^1} + \|\eta_x(t)\| + \|\theta_x(t)\| + \|f(t)\|).\end{aligned} \tag{25}$$

Differentiating (2) with respect to x and using Lemma 1 and Lemma 2, we deduce

$$\|v_{xt}(t)\| \leqslant C_2(\|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1}) + C_1\|f_\xi(t)\| \tag{26}$$

or

$$\|v_{3x}(t)\| \leqslant C_2(\|v(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \|v_{xt}(t)\| + \|\theta_x(t)\|_{H^1}) + C_1\|f_\xi(t)\|. \tag{27}$$

Similarly, differentiating (2) with respect to x twice, using Lemma 1 and Lemma 2 and the interpolation inequality, we arrive at

$$\|v_{xxt}(t)\| \leqslant C_2(\|u_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|f_\xi(t)\| + \|v_x(t)\|_{H^3} + \|f_{\xi\xi}(t)\|) \tag{28}$$

or

$$\|v_{4x}(t)\| \leqslant C_2(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_{xxt}(t)\| + \|f_\xi(t)\| + \|f_{\xi\xi}(t)\|). \tag{29}$$

We easily deduce from (3) and Lemma 1 and Lemma 2 that

$$\|\theta_t(t)\| \leqslant C_1(\|\theta_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|g(t)\|). \tag{30}$$

We differentiate (3) with respect to x, and use Lemma 1 and Lemma 2 to get

$$\|\theta_{xt}(t)\| \leqslant C_2(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^2} + \|g_\xi(t)\|) \tag{31}$$

or

$$\|\theta_{3x}(t)\| \leqslant C_2(\|u_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|\theta_{xt}(t)\| + \|v_x(t)\|_{H^1} + \|g_\xi(t)\|). \tag{32}$$

Differentiating (3) with respect to x twice, using Lemma 1 and Lemma 2 and the embedding theorem, we derive

$$\|\theta_{xxt}(t)\| \leqslant C_2(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|g_\xi(t)\| + \|g_{\xi\xi}(t)\| + \|\theta_x(t)\|_{H^3}), \tag{33}$$

or

$$\|\theta_{4x}(t)\| \leqslant C_2(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_{xxt}(t)\| + \|g_{\xi\xi}(t)\| + \|g_\xi(t)\|). \tag{34}$$

Differentiating (3) with respect to t, by (26), (28) and (30)–(31), we see that

$$\begin{aligned}\|v_{tt}(t)\| &\leqslant C_2(\|v_x(t)\|_{H^1} + \|u_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|v_{xt}(t)\| + \|v_{xxt}(t)\| + \|f_\xi(t)\| + \|f_t(t)\|) \quad (35)\\ &\leqslant C_2(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2}) + C_2(\|g(t)\| + \|g_\xi(t)\| + \|f_\xi(t)\| + \|f_{\xi\xi}(t)\| + \|f_t(t)\|). \quad (36)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\theta_{tt}(t)\| &\leqslant C_2(\|v_x(t)\| + \|u_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|v_{xt}(t)\| + \|\theta_x(t)\|_{H^2} + \|\theta_{xxt}(t)\| + \|g_\xi(t)\| + \|g_t(t)\|) \quad (37)\\ &\leqslant C_2(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2}) + C_2(\|g(t)\| + \|g_\xi(t)\| + \|f_\xi(t)\| + \|g_{\xi\xi}(t)\| + \|g_t(t)\|). \quad (38)\end{aligned}$$

Thus estimates (21)–(22) follow from (19), (26), (28), (31), (33), (35), (38) and (8)–(14).

Differentiating (2) with respect to t twice, multiplying the result by v_{tt} in $L^2(0,1)$, performing an integration by parts and using Young's inequality and using Lemma 1 and Lemma 2, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{tt}(t)\|^2 + \|v_{xtt}\|^2\\ &\leqslant C_2(\|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2) + C_1(\|f_{\xi\xi}(t)\|^2 + \|f_{\xi t}(t)\|^2 + \|f_{tt}(t)\|^2). \quad (39)\end{aligned}$$

Thus by (10)–(11), Lemma 1 and Lemma 2,

$$\|v_{tt}(t)\|^2 + \int_0^t \|v_{xtt}(s)\|^2 \,\mathrm{d}s \leqslant C_4 + C_2 \int_0^t \|\theta_{tt}(s)\|^2 \,\mathrm{d}s$$

which, along with (27), gives (23).

Differentiating (3) with respect to t twice, multiplying the resulting equation by θ_{tt} in $L^2(0,1)$, integrating by parts and using Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_{tt}^2 \,\mathrm{d}x\\ &= -\int_0^1 \left(\frac{\kappa\theta_x}{u}\right)_{tt} \theta_{xtt} \,\mathrm{d}x + \int_0^1 \left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)_{tt} v_x\theta_{tt} \,\mathrm{d}x\\ &\quad + 2\int_0^1 \left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right)_t v_{xt}\theta_{tt} \,\mathrm{d}x + \int_0^1 \left(-R\frac{\theta}{u} + \mu\frac{v_x}{u}\right) v_{xtt}\theta_{tt} \,\mathrm{d}x\\ &\quad + \int_0^1 (g_{\xi\xi}v^2 + 2g_{\xi t}v + g_\xi v_t + g_{tt})\theta_{tt} \,\mathrm{d}x\\ &:= \sum_{i=1}^5 A_i. \end{aligned}\tag{40}$$

Using Lemma 1 and Lemma 2 and the interpolation inequality, we get for any $\varepsilon > 0$

$$\begin{aligned}A_1 &\leqslant -(2C_1)^{-1}\|\theta_{xtt}(t)\|^2 + C_2(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xxt}(t)\|^2), \quad (41)\\ A_2 &\leqslant \varepsilon\|v_{xtt}(t)\|^2 + C_2\varepsilon^{-1}(\|\theta_{tt}(t)\|^2 + \|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2), \quad (42)\\ A_3 &\leqslant C_2\|v_{xt}(t)\|^{\frac{1}{2}}\|v_{xxt}(t)\|^{\frac{1}{2}}(\|v_x(t)\| + \|\theta_t(t)\| + \|v_{xt}(t)\|)\|\theta_{tt}(t)\|,\end{aligned}$$

which gives

$$\int_0^t A_3 \,\mathrm{d}s \leqslant \varepsilon \sup_{0\leqslant s\leqslant t} \|\theta_{tt}(s)\|^2 + \varepsilon \int_0^t \|v_{xxt}(s)\|^2 \,\mathrm{d}s + C_2\varepsilon^{-3}. \tag{43}$$

Analogously,

$$\begin{aligned}A_4 &\leqslant \varepsilon\|v_{xtt}(t)\|^2 + C_2\varepsilon^{-1}\|\theta_{tt}(t)\|^2, \quad (44)\\ A_5 &\leqslant \varepsilon\|\theta_{ttx}(t)\|^2 + C_2\varepsilon^{-1}(\|g_{\xi\xi}(t)\|^2 + \|g_{\xi t}(t)\|^2 + \|g_\xi(t)\|^2 + \|v_{xt}(t)\|^2 + \|g_{tt}(t)\|^2) \quad (45)\end{aligned}$$

Integrating (40) over $(0,t)$, using (41)–(45), (13)–(14) and Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}&\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{xtt}(s)\|^2 \,\mathrm{d}s\\ &\leqslant C_1\varepsilon\left(\sup_{0\leqslant s\leqslant t} \|\theta_{tt}(s)\|^2 + \int_0^t (\|v_{xxt}\|^2 + \|v_{xtt}\|^2)(s) \,\mathrm{d}s\right) + C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t (\|\theta_{tt}(s)\|^2 + \|\theta_{xxt}\|^2)(s) \,\mathrm{d}s,\end{aligned}$$

which, taking ε small enough and using (37) and Lemma 1, implies (24). The proof is now complete. □

Lemma 4 *Under the assumptions in Theorem 1, for any* $(u_0, v_0, \theta_0) \in H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$, *the following estimates hold for any* $\varepsilon > 0$

$$\|v_{xt}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2 \,\mathrm{d}s \leqslant C_3\varepsilon^{-6} + C_1\varepsilon^2 \int_0^t (\|\theta_{xxt}\|^2 + \|v_{xtt}\|^2)(s)\,\mathrm{d}s, \quad (46)$$

$$\|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(s)\|^2 \,\mathrm{d}s \leqslant C_3\varepsilon^{-6} + C_1\varepsilon^2 \int_0^t (\|v_{xxt}\|^2 + \|\theta_{xtt}\|^2)(s)\,\mathrm{d}s. \quad (47)$$

Proof: Differentiating (2) with respect to t and x, multiplying the result by v_{tx} in $L^2(0,1)$ and integrating by parts, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{xt}(t)\|^2 = \sigma_{tx}v_{xt}\big|_{x=0}^{x=1} - \int_0^1 \sigma_{xt}v_{xxt}\,\mathrm{d}x + \int_0^1 (f_{\xi\xi}vu + f_{\xi t}u + f_\xi v_x)v_{xt}\,\mathrm{d}x = \sum_{i=1}^3 D_i. \quad (48)$$

By Lemma 1, Lemma 2 and the interpolation inequality and Young's inequality, we get for any $\varepsilon > 0$

$$\begin{aligned} D_1 \leqslant{}& C_1(\|\theta_{xt}(t)\|_{L^\infty} + \|\theta_x(t)v_x(t)\|_{L^\infty} + \|\theta_t(t)v_x(t)\|_{L^\infty} + \|\theta(t)v_{xx}(t)\|_{L^\infty} \\ &+ \|\theta(t)u_x(t)v_x(t)\|_{L^\infty} + \|v_{xxt}(t)\|_{L^\infty} + \|v_{xx}(t)v_x(t)\|_{L^\infty} + \|v_{xt}(t)u_x(t)\|_{L^\infty} \\ &+ \|v_x^2(t)u_x(t)\|_{L^\infty})\|v_{xt}(t)\|_{L^\infty} \\ \leqslant{}& C_2\Big(\|v_x(t)\|_{H^2} + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|v_{xt}\| + \|\theta_{xt}\|^{\frac12}\|\theta_{xxt}(t)\|^{\frac12} + \|v_{xxt}\|^{\frac12}\|v_{3xt}(t)\|^{\frac12} \\ &+ \|v_{xt}\|^{\frac12}\|v_{xxt}(t)\|^{\frac12}\Big)\|v_{xt}\|^{\frac12}\|v_{xxt}(t)\|^{\frac12} \\ \leqslant{}& \varepsilon^2\|v_{xxt}(t)\|^2 + \varepsilon^2(\|\theta_{xxt}(t)\|^2 + \|v_{3xt}(t)\|^2) + C_2\varepsilon^{-6}(\|v_{xt}(t)\|^2 + \|v_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2), \end{aligned} \quad (49)$$

$$D_2 \leqslant -\frac{1}{(2C_1)}\|v_{xxt}(t)\|^2 + C_2(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|_{H^1}^2 + \|v_{xt}(t)\|^2 + \|u_x(t)\|^2), \quad (50)$$

$$D_3 \leqslant \varepsilon\|v_{xxt}(t)\|^2 + C_\varepsilon\|v_{xt}(t)\|^2 + C_\varepsilon(\|f_{\xi\xi}(t)\|^2 + \|f_{\xi t}(t)\|^2 + \|f_\xi(t)\|^2),$$

which, together with (10)–(11), (48)–(50) and (19)–(20), gives

$$\|v_{xt}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2\,\mathrm{d}s \leqslant C_1\varepsilon^2\int_0^t (\|\theta_{xxt}\|^2 + \|v_{3xt}(t)\|^2)(s)\,\mathrm{d}s + C_3\varepsilon^{-6}. \quad (51)$$

Differentiating (2) with respect to x and t, and using (19)–(20), we derive

$$\|v_{3xt}(t)\| \leqslant C_1\|v_{xxt}(t)\| + C_2(\|v_{xx}(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^2} + \|f_{\xi\xi}(t)\| + \|f_{\xi t}(t)\| + \|f_\xi(t)\|). \quad (52)$$

Then (46) follows from (10)–(11), (51)–(52), and (19)–(20).

Differentiating (3) with respect to x and t, multiplying the resulting equation by θ_{xt} in $L^2(0,1)$, and integrating by parts, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_{xt}^2\,\mathrm{d}x := \sum_{i=1}^4 E_i. \quad (53)$$

where

$$E_1 = \left(\frac{\kappa\theta_x}{\eta}\right)_{xt}\theta_{xt}\Big|_{x=0}^{x=1},$$

$$E_2 = -\int_0^1 \left(\frac{\kappa\theta_x}{\eta}\right)_{xt}\theta_{xxt}\,\mathrm{d}x,$$

$$E_3 = -\int_0^1 (\sigma v_x)_{xt}\theta_{xt}\,\mathrm{d}x, \quad E_4 = \int_0^1 g_{xt}\theta_{xt}\,\mathrm{d}x.$$

From (19)–(20) and the interpolation inequality and Young's inequality, we derive for any $\varepsilon > 0$

$$E_1 \leqslant \varepsilon^2(\|\theta_{xxt}(t)\|^2 + \|\theta_{3xt}(t)\|^2) + C_2\varepsilon^{-6}(\|v_x(t)\|_{H^2}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2), \quad (54)$$

$$E_2 \leqslant -\frac{1}{2C_1}\|\theta_{xxt}(t)\|^2 + C_2(\|v_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2), \quad (55)$$

$$E_3 \leqslant \varepsilon^2\|v_{xxt}(t)\|^2 + C_2\varepsilon^{-2}(\|v_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2 + \|v_{xt}(t)\|^2), \quad (56)$$

$$E_4 \leqslant \varepsilon^2\|\theta_{xxt}(t)\|^2 + C_2\varepsilon^{-2}(\|g_{\xi\xi}(t)\|^2 + \|g_{\xi t}(t)\|^2 + \|g_\xi(t)\|^2). \quad (57)$$

Differentiating (3) with respect to x and t, using the interpolation inequalities and Young's inequality and Lemma 1, Lemma 2 and Lemma 3, we conclude

$$\|\theta_{3xt}(t)\| \leqslant C_2(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_t(t)\|_{H^2} + \|\theta_{tt}(t)\|_{H^1} + \|v_{xt}(t)\|_{H^1} + \|g_{\xi\xi}(t)\| + \|g_{\xi t}(t)\| + \|g_\xi(t)\|). \quad (58)$$

Integrating (53) with respect to t over $(0,t)$ and using (54)–(58), (13)–(14) and (19)–(20), we can get (47). The proof is now complete. □

Lemma 5 *Under the assumptions in Theorem 1, for any* $(u_0, v_0, \theta_0) \in H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$, *the following estimates hold for any* $t > 0$

$$\|v_{tt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t (\|v_{xxt}\|^2 + \|v_{xtt}\|^2 + \|\theta_{xtt}\|^2 + \|\theta_{xxt}\|^2)(s)\,\mathrm{d}s \leqslant C_4, \quad (59)$$

$$\|u_{3x}(t)\|_{H^1}^2 + \|u_{xx}(t)\|_{W^{1,\infty}}^2 + \int_0^t (\|u_{3x}\|_{H^1}^2 + \|u_{xx}\|_{W^{1,\infty}}^2)(s)\,\mathrm{d}s \leqslant C_4, \quad (60)$$

$$\|v_{3x}(t)\|_{H^1}^2 + \|v_{xx}(t)\|_{W^{1,\infty}}^2 + \|\theta_{3x}(t)\|^2 + \|\theta_{xx}\|_{W^{1,\infty}}^2 + \|u_{4x}(t)\|^2 + \|v_{xxt}(t)\|^2 + \|\theta_{xxt}(t)\|^2 + \int_0^t (\|v_{tt}\|^2 + \|\theta_{tt}\|^2 + \|v_{xx}\|_{W^{2,\infty}}^2 + \|\theta_{xx}\|_{W^{2,\infty}}^2 + \|\theta_{xxt}\|_{H^1}^2 + \|v_{xxt}\|_{H^1}^2 + \|\theta_{xt}\|_{W^{1,\infty}}^2 + \|v_{xt}\|_{W^{1,\infty}}^2 + \|u_{3xt}\|_{H^1}^2)(s)\,\mathrm{d}s \leqslant C_4, \quad (61)$$

$$\int_0^t (\|v_{4x}\|_{H^1}^2 + \|\theta_{4x}\|_{H^1}^2)(s)\,\mathrm{d}s \leqslant C_4. \quad (62)$$

Proof: We add up (46) and (47) and take $\varepsilon >$ so small to get

$$\|v_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t (\|v_{xxt}\|^2 + \|\theta_{xxt}\|^2)(s)\,\mathrm{d}s \leqslant C_3\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t (\|v_{xtt}\|^2 + \|\theta_{ttx}\|^2)(s)\,\mathrm{d}s. \quad (63)$$

Multiplying (23) and (24) by ε and $\varepsilon^{\frac{3}{2}}$, respectively, then adding the result to (63), picking ε small enough and using (19)–(20), we get (59).

Differentiating (2) with respect to x twice and using (1), we deduce

$$\mu\left(\frac{u_{3x}}{u}\right)_t + R\frac{\theta u_{3x}}{u^2} = K_1(x,t), \quad (64)$$

where

$$K_1(x,t) = v_{xxt} + \mu\frac{v_{3x}u_x}{u^2} + \mu\frac{u_{xx}v_{xx}}{u^2} + K_x - 2\mu\frac{u_{xx}v_x u_x}{u^3} + 2\frac{\theta u_{xx}u_x}{u^3} - \frac{\theta_x u_{xx}}{u^2}, \quad (65)$$

$$K(x,t) = R\frac{\theta_{xx}}{u} - 2R\frac{\theta_x u_x}{u^2} + 2R\frac{\theta u_x^2}{u^3} + 2\mu\frac{v_{xx}u_x}{u^2} - 2\mu\frac{v_x u_x^2}{u^3} - f_\xi u.$$

From the interpolation inequalities, Young's inequality, (19)–(20) and Lemma 3, we derive

$$\|K_1(t)\| \leqslant C_2(\|\theta_x(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|f_{\xi\xi}(t)\| + \|f_\xi(t)\|), \quad (66)$$

which, combined with (9)–(10), (19)–(20) and (59), gives

$$\int_0^t \|K_1(s)\|^2\,\mathrm{d}s \leqslant C_1, \ \ \forall t > 0. \quad (67)$$

Now multiplying (64) by u_{3x}/u and integrating the result over $(0,1)$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{u_{3x}}{u}(t)\right\|^2 + C_1^{-1}\left\|\frac{u_{3x}}{u}(t)\right\|^2 \leqslant C_1\|K_1(t)\|^2, \quad (68)$$

which, together with (67), implies

$$\|u_{3x}(t)\|^2 + \int_0^t \|u_{3x}(s)\|^2\,\mathrm{d}s \leqslant C_4, \ \ \forall t > 0. \quad (69)$$

By (27), (29), (32), (34), (59) and (19)–(20), and using the interpolation inequality, we obtain for any $t > 0$,

$$\|v_{3x}(t)\|^2 + \|\theta_{3x}(t)\|^2 + \|v_{xx}(t)\|_{L^\infty}^2 + \|\theta_{xx}(t)\|_{L^\infty}^2 + \int_0^t (\|v_{4x}\|_{H^1}^2 + \|\theta_{3x}\|_{H^1}^2 + \|v_{xx}\|_{W^{1,\infty}}^2 + \|\theta_{xx}\|_{W^{1,\infty}}^2)(s)\,\mathrm{d}s \leqslant C_4. \quad (70)$$

Differentiating (2)–(3) with respect to t, using (32), (59) and (19)–(20), we deduce that for any $t > 0$,

$$\|v_{xxt}(t)\| \leqslant C_1\|v_{tt}(t)\| + C_2(\|v_x(t)\|_{H^1} + \|v_{xt}(t)\| + \|\theta_t(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|f_\xi(t)\| + \|f_t(t)\|) \leqslant C_4, \quad (71)$$

$$\|\theta_{xxt}(t)\| \leqslant C_1\|\theta_{tt}(t)\| + C_2(\|v_x(t)\|_{H^1} + \|v_{xt}(t)\| + \|\theta_t(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|g_\xi(t)\| + \|g_t(t)\|) \leqslant C_4 \quad (72)$$

which, together with (29), (34), (59) and (19)–(20), yields

$$\|v_{4x}(t)\|^2+\|\theta_{4x}(t)\|^2+\int_0^t(\|\theta_{xxt}\|^2+\|\theta_{4x}\|^2 + \|v_{xxt}\|^2+\|v_{4x}\|^2)(s)\,\mathrm{d}s\leqslant C_4,\ \forall\, t>0. \quad (73)$$

Performing the interpolation inequality, (70) and (73), we get for any $t>0$,

$$\|v_{3x}(t)\|_{L^\infty}^2+\|\theta_{3x}(t)\|_{L^\infty}^2 +\int_0^t(\|v_{3x}\|_{L^\infty}^2+\|\theta_{3x}\|_{L^\infty}^2)(s)\,\mathrm{d}s\leqslant C_4. \quad (74)$$

We infer from (36), (38) and (74) that for any $t>0$,

$$\int_0^t(\|v_{tt}\|^2+\|\theta_{tt}\|^2)(s)\,\mathrm{d}s\leqslant C_4, \quad (75)$$

which, along with (52) and (58)–(59), gives

$$\int_0^t(\|v_{3xt}\|^2+\|\theta_{3xt}\|^2)(s)\,\mathrm{d}s\leqslant C_4,\ \forall t>0. \quad (76)$$

Differentiating (64) with respect to x, we arrive at

$$\mu\left(\frac{u_{4x}}{u}\right)_t+R\frac{\theta u_{4x}}{u^2}=K_2(x,t) \quad (77)$$

with

$$K_2(x,t)=K_{1x}-R\frac{\theta_x u_{3x}}{u^2}+2R\frac{\theta u_{3x}u_x}{u^3} +\mu\left(\frac{u_{3x}u_x}{u^2}\right)_t.$$

Using the embedding theorem, (19)–(20) and (59), we derive that for any $t>0$

$$\|K_2(t)\|\leqslant C_1\|v_{xxt}(t)\|+C_4(\|\theta_x(t)\|_{H^3} +\|u_x(t)\|_{H^2}+\|v_x(t)\|_{H^3}+\|f_{3\xi}(t)\| +\|f_{\xi\xi}(t)\|+\|f_\xi(t)\|),$$

which, together with (9)–(11), (19)–(20), (59) and (73), implies

$$\int_0^t\|K_2(s)\|^2\,\mathrm{d}s\leqslant C_4,\ \forall t>0. \quad (78)$$

Multiplying (77) by u_{4x}/u in $L^2(0,1)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{u_{4x}}{u}(t)\right\|^2+C_1^{-1}\left\|\frac{u_{4x}}{u}(t)\right\|^2\leqslant C_1\|K_2(t)\|^2, \quad (79)$$

which, along with (78), gives

$$\|u_{4x}(t)\|^2+\int_0^t\|u_{4x}(s)\|^2\,\mathrm{d}s\leqslant C_4,\quad \forall\, t>0. \quad (80)$$

Differentiating (19) with respect to x three times, using (19)–(20) and the interpolation inequalities, we infer

$$\|v_{5x}(t)\|\leqslant C_1\|v_{3xt}(t)\|+C_2(\|u_x(t)\|_{H^3} +\|v_x(t)\|_{H^3}+\|\theta_x(t)\|_{H^3}+\|f_{3\xi}(t)\| +\|f_{\xi\xi}(t)\|+\|f_\xi(t)\|),$$

which, together with (1), (9)–(11), (76) and (80), yields

$$\int_0^t(\|v_{5x}\|^2+\|u_{3xt}\|_{H^1}^2)(s)\,\mathrm{d}s\leqslant C_4,\ \forall t>0. \quad (81)$$

Similarly, from (3), we derive

$$\|\theta_{5x}(t)\|\leqslant C_4(\|u_x(t)\|_{H^3}+\|v_x(t)\|_{H^3}+\|\theta_x(t)\|_{H^3} +\|\theta_{xxt}(t)\|_{H^1}+\|g_{3\xi}(t)\|+\|g_{\xi\xi}(t)\| +\|g_\xi(t)\|). \quad (82)$$

From (13)–(14), (80), (73) and (76), we conclude for any $t>0$

$$\int_0^t\|\theta_{5x}(s)\|^2\,\mathrm{d}s\leqslant C_4, \quad (83)$$

which, together with (81) and (83), gives

$$\int_0^t(\|v_{xx}\|_{W^{2,\infty}}^2+\|\theta_{xx}\|_{W^{2,\infty}}^2)(s)\,\mathrm{d}s\leqslant C_4,\ \forall\, t>0. \quad (84)$$

Finally, using all the previous estimates and the interpolation inequality, we can easily derive the desired estimates (60)–(62). The proof is complete. □

LARGE-TIME BEHAVIOUR IN $H^4(0,1)\times H_0^4(0,1)\times H^4(0,1)$

In this section, we shall derive the large-time behaviour in $H^4(0,1)\times H_0^4(0,1)\times H^4(0,1)$. To begin with, we need a differential inequality in next lemma.

Lemma 6 *Let T be given with $0<T\leqslant+\infty$. Suppose that y and h are nonnegative continuous functions defined on $[0,T]$ and satisfy the following conditions:*

$$\frac{\mathrm{d}y}{\mathrm{d}t}\leqslant A_1y^2(t)+A_2+h(t),$$
$$\int_0^T y(s)\,\mathrm{d}s\leqslant A_3,\quad \int_0^T h(s)\,\mathrm{d}s\leqslant A_4,$$

where A_1,A_2,A_3,A_4 are given nonnegative constants. Then for any $r>0$, with $0<r<T$,

$$y(t+r)\leqslant\left(\frac{A_3}{r}+A_2r+A_4\right)\cdot\mathrm{e}^{A_1A_3}.$$

Furthermore, if $T=+\infty$, then

$$\lim_{t\to+\infty} y(t)=0.$$

Proof: See, e.g., Ref. 16. □

Lemma 7 *Under the assumptions in Lemma 1, we have*

$$\lim_{t\to+\infty}\|u(t)-\overline{u}\|_{H^1}=0,\ \lim_{t\to+\infty}\|v(t)\|_{H^1}=0, \tag{85}$$
$$\lim_{t\to+\infty}\|\theta(t)-\overline{\theta}\|_{H^1}=0$$

where $\overline{u}=\int_0^1 u\,\mathrm{d}x$ and $\overline{\theta}=\int_0^1\theta(y,t)\,\mathrm{d}y$.

Proof: See, e.g., Ref. 15. □

Lemma 8 *Under the assumptions in Lemma 2, we have*

$$\lim_{t\to+\infty}\|u(t)-\overline{u}\|_{H^2}=0,\ \lim_{t\to+\infty}\|v(t)\|_{H^2}=0, \tag{86}$$
$$\lim_{t\to+\infty}\|\theta(t)-\overline{\theta}\|_{H^2}=0$$

where $\overline{u}=\int_0^1 u_0\,\mathrm{d}x$ and $\overline{\theta}=\int_0^1\theta(y,t)\,\mathrm{d}y$.

Proof: See, e.g., Ref. 15. □

Lemma 9 *Under the assumptions in Theorem 1, we have*

$$\lim_{t\to+\infty}\|u(t)-\overline{u}\|_{H^4}=0, \tag{87}$$

where $\overline{u}=\int_0^1 u_0\,\mathrm{d}x$.

Proof: In (68), we have deduced

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{u_{3x}}{\eta}(t)\right\|^2+C_1^{-1}\left\|\frac{u_{3x}}{\eta}(t)\right\|^2\leqslant\|K_1(t)\|^2$$

where

$$\|K_1(t)\|\leqslant C_2(\|\theta_x(t)\|_{H^2}+\|u_x(t)\|_{H^1}+\|v_x(t)\|_{H^2}+\|f_{\xi\xi}(t)\|+\|f_\xi(t)\|).$$

Using (19)–(20) and Lemma 6, we get

$$\lim_{t\to+\infty}\|u_{3x}(t)\|^2=0. \tag{88}$$

Recalling (78)–(79) and Lemma 6, we obtain

$$\lim_{t\to+\infty}\|u_{4x}(t)\|^2=0,$$

which, together with (85), (88) and Poincaré's inequality, yields (87). The proof is complete. □

Lemma 10 *Under the assumptions in Theorem 1, we have*

$$\lim_{t\to+\infty}\|v(t)\|_{H^4}=0. \tag{89}$$

Proof: Differentiating (2) with respect to x and t, multiplying the result by v_{xt} in $L^2(0,1)$ and using (19)–(20) and Lemma 5, we derive for any $\varepsilon>0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v_{xt}(t)\|^2+\|v_{xxt}(t)\|^2\\&\leqslant\varepsilon(\|v_{3xt}(t)\|^2+\|\theta_{xxt}(t)\|^2)+C_4(\|v_x(t)\|_{H^2}^2\\&\quad+\|\theta_t(t)\|_{H^1}^2+\|v_{xt}(t)\|^2+\|u_x(t)\|^2+\|f_{\xi\xi}(t)\|^2\\&\quad+\|f_{\xi t}(t)\|^2+\|f_\xi(t)\|^2).\end{aligned} \tag{90}$$

Using (90), (9)–(11), (19)–(20), Lemma 5 and Lemma 6, we obtain

$$\lim_{t\to+\infty}\|v_{xt}(t)\|^2=0. \tag{91}$$

Now we claim that

$$\lim_{t\to+\infty}\|f_\xi(t)\|^2=0. \tag{92}$$

In fact,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\xi(t)\|^2&=2\int_0^1 f_\xi\frac{\mathrm{d}f_\xi}{\mathrm{d}t}\,\mathrm{d}x\\&\leqslant C_2(\|f_\xi(t)\|^2+\|f_{\xi t}(t)\|^2+\|f_{\xi\xi}(t)\|^2),\end{aligned}$$

which, together with (9)–(11) and Lemma 6, gives (92).
Similarly, we can get

$$\lim_{t\to+\infty}\|g_\xi(t)\|^2=0,\ \lim_{t\to+\infty}\|g_t(t)\|^2=0, \tag{93}$$
$$\lim_{t\to+\infty}\|f_{\xi\xi}(t)\|^2=0,\ \lim_{t\to+\infty}\|g_{\xi\xi}(t)\|^2=0.$$

Using (26), (85)–(86) and (91)–(92), we have

$$\lim_{t\to+\infty}\|v_{3x}(t)\|=0. \tag{94}$$

By (19)–(20) and Lemma 5, and using the interpolation inequality, we obtain

$$\|p_{xxt}(t)\|\leqslant C_2(\|v_x(t)\|_{H^2}+\|u_x(t)\|_{H^2}+\|\theta_t(t)\|_{H^2}+\|\theta_x(t)\|_{H^2}). \tag{95}$$

Differentiating (2) with respect to t once and x twice, multiplying the resulting by v_{xxt} in $L^2(0,1)$ and using Young's inequality and (19)–(20), we derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v_{xxt}(t)\|^2+\|v_{3xt}(t)\|^2\\&\leqslant C_1\|p_{xxt}(t)\|^2+C_2(\|v_{xt}(t)\|_{H^2}^2+\|v_x(t)\|_{H^2}^2\\&\quad+\|u_x(t)\|_{H^2}^2+\|f_{3\xi}(t)\|^2+\|f_{\xi\xi t}(t)\|^2+\|f_{\xi\xi}(t)\|^2\\&\quad+\|f_{\xi t}(t)\|^2+\|f_\xi(t)\|^2),\end{aligned}$$

which, together with (19)–(20), (9)–(11), Lemma 5 and Lemma 6, gives

$$\lim_{t\to+\infty} \|v_{xxt}(t)\|^2 = 0. \tag{96}$$

Differentiating (3) with respect to x and t, multiplying the result by θ_{xt} in $L^2(0,1)$ and using (19)–(20) and Lemma 5, we deduce for any $\varepsilon > 0$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{xt}(t)\|^2 + \|\theta_{xxt}(t)\|^2\\ &\leqslant \varepsilon(\|\theta_{3xt}(t)\|^2 + \|v_{xxt}(t)\|^2) + C_\varepsilon(\|v_x(t)\|^2_{H^2}\\ &+ \|\theta_x(t)\|^2_{H^2} + \|\theta_t(t)\|^2_{H^1} + \|u_x(t)\|^2_{H^1} + \|v_{xt}(t)\|^2\\ &+ \|g_{\xi\xi}(t)\|^2 + \|g_{\xi t}(t)\|^2 + \|g_\xi(t)\|^2),\end{aligned}$$

which, combined with (13)–(14), Lemma 5 and Lemma 6, implies

$$\lim_{t\to+\infty} \|\theta_{xt}(t)\|^2 = 0. \tag{97}$$

By (32), (97) and (85)–(86), we see that

$$\lim_{t\to+\infty} \|\theta_{3x}(t)\| = 0. \tag{98}$$

Thus by (29), (85)–(86), (96), (93) and (98), we get

$$\lim_{t\to+\infty} \|v_{4x}(t)\| = 0,$$

which, together with (94) and (85)–(86), yields (89). The proof is now complete. □

Lemma 11 *Under the assumptions in Theorem 1, we have*

$$\lim_{t\to+\infty} \|\theta(t) - \overline{\theta}\|_{H^4} = 0, \tag{99}$$

where $\overline{\theta} = \int_0^1 \theta(x,t)\,\mathrm{d}x$.

Proof: In Ref. 15, we have deduced

$$\lim_{t\to+\infty} \|\theta_t(t)\|^2 = 0. \tag{100}$$

In Lemma 1, we have obtained for any $\varepsilon > 0$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{tt}(t)\|^2 + \|\theta_{xxt}(t)\|^2\\ &\leqslant \varepsilon(\|\theta_{xtt}(t)\|^2 + \|v_{xtt}(t)\|^2) + C_2(\|v_x(t)\|^2_{H^1}\\ &+ \|\theta_{tt}(t)\|^2 + \|\theta_t(t)\|^2_{H^1} + \|v_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2\\ &+ \|g_{\xi\xi}(t)\|^2 + \|g_{\xi t}(t)\|^2 + \|g_\xi(t)\|^2 + \|v_{xt}(t)\|^2\\ &+ \|g_{tt}(t)\|^2),\end{aligned}$$

which, together with (13)–(14), (19)–(20), Lemma 5 and Lemma 6, gives

$$\lim_{t\to+\infty} \|\theta_{tt}(t)\|^2 = 0. \tag{101}$$

We infer from (3) that

$$\begin{aligned}\|\theta_{xxt}(t)\| \leqslant C_2(&\|\theta_{tt}(t)\| + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^1}\\ &+ \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|\theta_x(t)\|_{H^2}\\ &+ \|g_\xi(t)\| + \|g_t(t)\|)\end{aligned}$$

whence, by (37), (93), (88)–(89), (97), (100) and (85)–(86),

$$\lim_{t\to+\infty} \|\theta_{xxt}(t)\| = 0,$$

which, combined with (85)–(86) and (34), yields

$$\lim_{t\to+\infty} \|\theta_{4x}(t)\| = 0. \tag{102}$$

Thus (99) follows from (98) and (102). The proof is complete. □

PROOF OF THEOREM 1

Combining Lemma 5, Lemma 9, Lemma 10 and Lemma 11, we can complete the proof of Theorem 1.

Acknowledgements: B. Feng was supported by the Doctoral Innovational Fund of Donghua University with contract number BC201138. X. Yang was supported by the Young Teacher Research Fund of Henan Normal University (No. qd12104) and Innovation Scientists and Technicians Troop Construction Projects of Henan Province (No. 114200510011).

REFERENCES

1. Kanel Y (1968) On a model system of equations of one-dimensional gas motion. *Differ Equat* **4**, 374–80.
2. Itaya N (1976) A survey on the generalized Burgers equation with pressure model term. *J Math Kyoto Univ* **16**, 223–40.
3. Kazhikhov AV (1975) Correctness “in the large” of initial-boundary-value problem for model system of equations of a viscous gas. *Din Sploshnoi Sredy* **21**, 18–47.
4. Kzhikhov AV, Nikolaev VB (1979) On the correctness of boundary value problems for the equations of a viscous gas with a non-monotonic function of state. *Chislennye Metody Mekh Sploshnoi Sredy* **10**, 77–84.
5. Kazhikhov AV, Nikolaev VB (1979) On the theory of the Navier-Stokes equations of a viscous gas with nonmonotone state function. *Soviet Math Dokl* **20**, 583–5.
6. Kazhikhov AV, Shelukhin VV (1977) Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *J Appl Math Mech* **41**, 272–83.
7. Mucha PB (2001) Compressible Navier-Stokes system in 1-D. *Math Meth Appl Sci* **24**, 607–22.
8. Zhang T, Fang D (2006) Global behavior of compressible Navier-Stokes equations with degenerate viscosity coefficient. *Arch Ration Mech Anal* **182**, 223–53.

9. Qin Y, Zhao Y (2008) Global existence and asymptotic behavior of compressible Navier-Stokes equations for a one-dimensional isothermal viscous gas. *Math Model Meth Appl Sci* **18**, 1183–408.
10. Qin Y, Wu Y, Liu F (2007) On the Cauchy problem for a one-dimensional compressible viscous polytropic ideal gas. *Port Math* **64**, 87–126.
11. Zheng S, Qin Y (2001) Maximal attractor for the system of one-dimensional polytropic viscous ideal gas. *Q Appl Math* **59**, 579–99.
12. Jiang S (1994) On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas. *Math Z* **216**, 317–36.
13. Jiang S (1998) Large-time behavior of solutions to the equations of a viscous polytropic ideal gas. *Ann Mat Pura Appl* **175**, 253–75.
14. Qin Y (2008) *Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors*, Birkhäuser, Berlin.
15. Qin Y, Yu X (2009) Global existence and asymptotic behavior for the compressible Navier-Stokes equations with a non-autonomous external force and a heat source. *Math Meth Appl Sci* **32**, 1011–40.
16. Shen W, Zheng S (1993) On the coupled Cahn-Hilliard equations. *Comm Part Differ Equat* **18**, 701–27.