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An n-dimensional mixed-type additive and quadratic functional equation and its stability

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ABSTRACT: We determine the general solution of the n-dimensional mixed-type additive and quadratic functional equation, $2f(\sum_{i=1}^{n} x_i) + \sum_{1\leqslant i,j\leqslant n, i\neq j} f(x_i - x_j) = (n+1)\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} f(-x_i)$, where $n > 1$, and investigate its general stability.

KEYWORDS: polynomial extension, Banach space, inequality

INTRODUCTION

A functional equation is an equation in which the unknowns are functions. One widely studied functional equation is the additive functional equation, $f(x+y) = f(x)+f(y)$, for which it has been proved that the only continuous solution on a Banach space is a linear function (see, for example, pp. 36–7 of Ref. 1). In 1940, Ulam posed the following question concerning the stability of homomorphisms (see Ch. 6 of Ref. 2). Let G_1 be a group and let G_2 be a metric group with metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies the inequality

$$d\left(f(xy), f(x)f(y)\right) < \delta \quad \forall x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d\left(f(x), H(x)\right) < \varepsilon \quad \forall x \in G_1?$$

If the answer to the above question is affirmative, the functional equation for the homomorphisms will be called *stable*.

In 1941, Hyers[3] gave the first affirmative answer to the question of Ulam for the case where G_1 and G_2 are Banach spaces. Assume that E_1 and E_2 are Banach spaces. If a function $f : E_1 \to E_2$ satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$ for some $\varepsilon \geqslant 0$ for all $x, y \in E_1$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$ and $a : E_1 \to E_2$ is the unique additive function such that $\|f(x) - a(x)\| \leqslant \varepsilon$ for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then a is linear.

A generalization of Hyers' theorem was given by Aoki[4]. Later, Rassias[5] published the following stability theorem. If a function $f : E_1 \to E_2$ between Banach spaces satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\left(\|x\|^p + \|y\|^p\right)$$

for some $\theta \geqslant 0$, $0 \leqslant p < 1$ for all $x, y \in E_1$, then there exists an additive function $a : E_1 \to E_2$ such that

$$\|f(x) - a(x)\| \leqslant \frac{2\theta}{2 - 2^p}\|x\|^p$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then a is linear. Since then, the stability problem of several functional equations have been extensively investigated by a number of authors[6–8].

For real vector spaces X and Y, a function $f : X \to Y$ will be called *additive* if

$$f(x+y) = f(x) + f(y) \quad \forall x, y \in X \qquad (1)$$

and will be called *quadratic* if

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \forall x, y \in X. \qquad (2)$$

We will thus call a function $f : X \to Y$ *mixed-type additive and quadratic* if there exists an additive function $a : X \to Y$ and a quadratic function $q : X \to Y$ such that

$$f(x) = a(x) + q(x) \quad \forall x \in X.$$

Mixed-type functional equations have recently been studied by quite a few researchers[8,9].

In this paper, we will study the general solution of an n-dimensional mixed-type additive and quadratic functional equation and then investigate its general stability.

THE GENERAL SOLUTION

Theorem 1 *Let $n > 1$ be an integer, and let X and Y be vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} f(x_i - x_j) = (n+1)\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} f(-x_i) \quad (3)$$

for all $x_1, x_2, \dots, x_n \in X$ if and only if there exist an additive function $a : X \to Y$ and a quadratic function $q : X \to Y$ such that

$$f(x) = a(x) + q(x) \quad \forall x \in X. \quad (4)$$

Proof: Suppose a function $f : X \to Y$ satisfies (3). Putting $(x_1, x_2, \dots, x_n) = (0, \dots, 0)$ in (3), we get $f(0) = 0$. Setting $(x_1, x_2, \dots, x_n) = (x, y, 0, \dots, 0)$ in (3) yields

$$\begin{aligned}&2f(x+y) + f(x-y) + f(y-x)\\ &\quad + (n-2)\left(f(x) + f(y) + f(-x) + f(-y)\right)\\ &= (n+1)\left(f(x) + f(y)\right) + (n-1)\left(f(-x) + f(-y)\right),\end{aligned}$$

which simplifies to

$$\begin{aligned}&2f(x+y) + f(x-y) + f(y-x)\\ &\quad = 3f(x) + 3f(y) + f(-x) + f(-y) \quad (5)\end{aligned}$$

for all $x, y \in X$. We define the even part and the odd part of function f by

$$f_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2}, \; f_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2}.$$

Replacing x and y with $-x$ and $-y$ in (5),

$$\begin{aligned}&2f(-x-y) + f(-x+y) + f(-y+x)\\ &\quad = 3f(-x) + 3f(-y) + f(x) + f(y). \quad (6)\end{aligned}$$

Taking half the sum and half the difference of (5) and (6), we immediately see that f_{e} and f_{o} satisfy the classical quadratic functional equation and the additive functional equation, respectively, i.e.,

$$\begin{gathered}2f_{\mathrm{e}}(x+y) + 2f_{\mathrm{e}}(x-y) = 4f_{\mathrm{e}}(x) + 4f_{\mathrm{e}}(y),\\ 2f_{\mathrm{o}}(x+y) = 2f_{\mathrm{o}}(x) + 2f_{\mathrm{o}}(y).\end{gathered}$$

Now assume that there exist an additive function $a : X \to Y$ and a quadratic function $q : X \to Y$ such that $f(x) = a(x) + q(x)$ for all $x \in X$. Then it follows that $q(x) = B(x,x)$ for some biadditive mapping $B : X^2 \to Y$ (see pp. 89–91 of Ref. 1). We want to show that f satisfies (3). Considering the left-hand side of (3) we obtain

$$\begin{aligned}&2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} f(x_i - x_j)\\ &= 2a\left(\sum_{i=1}^{n} x_i\right) + 2B\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i\right)\\ &\quad + \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} a(x_i - x_j) + \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} B(x_i - x_j, x_i - x_j).\end{aligned} \quad (7)$$

By the additivity of a,

$$a\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} a(x_i). \quad (8)$$

Since $a(-x) = -a(x)$ for all $x \in X$,

$$\sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} a(x_i - x_j) = 0. \quad (9)$$

Since $B(x,x)$ is symmetric and biadditive,

$$B\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} B(x_i, x_j) \quad (10)$$

and

$$\begin{aligned}&\sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} B(x_i - x_j, x_i - x_j)\\ &= \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \left(B(x_i, x_i) - 2B(x_i, x_j) + B(x_j, x_j)\right)\\ &= 2(n-1)\sum_{i=1}^{n} B(x_i, x_i) - 2\sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} B(x_i, x_j)\\ &= 2n\sum_{i=1}^{n} B(x_i, x_i) - 2\sum_{i=1}^{n}\sum_{j=1}^{n} B(x_i, x_j). \quad (11)\end{aligned}$$

Substituting (8), (9), (10) and (11) in (7), we obtain

$$2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} f(x_i - x_j)$$
$$= 2\sum_{i=1}^{n} a(x_i) + 2n\sum_{i=1}^{n} B(x_i, x_i)$$
$$= (n+1)\sum_{i=1}^{n} (a(x_i) + B(x_i, x_i))$$
$$+ (n-1)\sum_{i=1}^{n} (a(-x_i) + B(-x_i, -x_i))$$
$$= (n+1)\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} f(-x_i).$$

which completes the proof. □

GENERAL STABILITY

In this section, the stability of (3) will be observed. We define

$$Df(x_1,\dots,x_n) = 2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} f(x_i - x_j) - (n+1)\sum_{i=1}^{n} f(x_i) - (n-1)\sum_{i=1}^{n} f(-x_i). \quad (12)$$

Theorem 2 *Let X be a real vector space. Let Y be a Banach space and let $n > 1$ be an integer. Let $\phi : X^n \to [0,\infty)$ be an even function with respect to each independent variable. Define $\varphi(x) = \phi(x, x, 0, \dots, 0)$ for all $x \in X$. Define the following two conditions:*
(i) $\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x)$ converges for all $x \in X$ and $\lim_{s\to\infty} 2^{-s}\phi(2^s x_1, \dots, 2^s x_n) = 0$, for all $x_i \in X$, with $i = 1, \dots, n$.
(ii) $\sum_{i=0}^{\infty} 4^i\varphi(2^{-i} x)$ converges for all $x \in X$, and $\lim_{s\to\infty} 4^s\phi(2^{-s} x_1, \dots, 2^{-s} x_n) = 0$, for all $x_i \in X$, with $i = 1, \dots, n$.
If a function $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \dots, x_n)\| \leqslant \phi(x_1, x_2, \dots, x_n) \quad (13)$$

for all $x_1, x_2, \dots, x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies (3) and, for all $x \in X$,

$$\|f(x) - T(x)\| \leqslant \left(\frac{1+\sigma}{12}\right)\varphi(0) + \frac{1}{2}\sum_{i=(1-\sigma)/2}^{\infty}\left(\frac{1+2^{1+\sigma i}}{4^{1+\sigma i}}\right)\varphi(2^{\sigma i}x) \quad (14)$$

where $\sigma = 1$ if condition (i) holds and $\sigma = -1$ if condition (ii) holds. The function T is given by

$$T(x) = \lim_{s\to\infty} 4^{-\sigma s} f_{\mathrm{e}}(2^{\sigma s}x) + 2^{-\sigma s} f_{\mathrm{o}}(2^{\sigma s}x)$$

for all $x \in X$.

Proof: Putting $(x_1, \dots, x_n) = (0, \dots, 0)$ in (13) yields

$$\|f(0)\| \leqslant \frac{\varphi(0)}{(n+2)(n-1)}. \quad (15)$$

First, we will prove the case when a function ϕ satisfies condition (i). We put $(x_1, x_2, \dots, x_n) = (x, x, 0, \dots, 0)$ in (13) to obtain that

$$\left\| f(2x) - 3f(x) - f(-x) - \frac{A}{2} f(0) \right\| \leqslant \frac{\varphi(x)}{2}, \quad (16)$$

where $A = n^2 + n - 8$. Replacing x by $-x$ in (16)

$$\left\| f(-2x) - 3f(-x) - f(x) - \frac{A}{2} f(0) \right\| \leqslant \frac{\varphi(-x)}{2} = \frac{\varphi(x)}{2}. \quad (17)$$

We apply the triangle inequality with (16) and (17) to get that

$$\left\| f_{\mathrm{e}}(2x) - 4f_{\mathrm{e}}(x) - \frac{A}{2} f(0) \right\| \leqslant \frac{\varphi(x)}{2} \quad (18)$$

and

$$\|f_{\mathrm{o}}(2x) - 2f_{\mathrm{o}}(x)\| \leqslant \frac{\varphi(x)}{2}. \quad (19)$$

Define a function $g_{\mathrm{e}} : X \to Y$ by $g_{\mathrm{e}}(x) = f_{\mathrm{e}}(x) + Af(0)/6$ for all $x \in X$. Note that

$$\left\| g_{\mathrm{e}}(x) - 4^{-1} g_{\mathrm{e}}(2x) \right\| \leqslant \frac{\varphi(x)}{8}. \quad (20)$$

Then for each positive integer s, we obtain

$$\left\| g_{\mathrm{e}}(x) - 4^{-s} g_{\mathrm{e}}(2^s x) \right\|$$
$$= \left\| \sum_{i=0}^{s-1} \left(4^{-i} g_{\mathrm{e}}(2^i x) - 4^{-(i+1)} g_{\mathrm{e}}(2^{(i+1)} x)\right) \right\|$$
$$\leqslant \sum_{i=0}^{s-1} 4^{-i} \left\| g_{\mathrm{e}}(2^i x) - 4^{-1} g_{\mathrm{e}}(2\cdot 2^i x) \right\|$$
$$\leqslant \frac{1}{8}\sum_{i=0}^{s-1} 4^{-i}\varphi(2^i x).$$

Similarly, for each positive integer s,

$$\left\| f_{\mathrm{o}}(x) - 2^{-s} f_{\mathrm{o}}(2^s x) \right\| \leqslant \frac{1}{4}\sum_{i=0}^{s-1} 2^{-i}\varphi(2^i x).$$

Consider the sequence $\{4^{-s}g_{\mathrm{e}}(2^s x)\}_{s=0}^{\infty}$. For every positive integer t, we have

$$\begin{aligned}\left\|4^{-s}g_{\mathrm{e}}(2^s x)-4^{-(s+t)}g_{\mathrm{e}}(2^{s+t}x)\right\| \\ &= 4^{-s}\left\|g_{\mathrm{e}}(2^s x)-4^{-t}g_{\mathrm{e}}(2^t\cdot 2^s x)\right\| \\ &\leqslant \frac{4^{-s}}{8}\sum_{i=0}^{t-1}4^{-i}\varphi(2^i\cdot 2^s x) \\ &\leqslant \frac{1}{8}\sum_{i=0}^{\infty}4^{-(i+s)}\varphi(2^{i+s}x).\end{aligned}$$

From condition (i), the convergence of $\sum_{i=0}^{\infty}2^{-i}\varphi(2^i x)$ implies that $\sum_{i=0}^{\infty}4^{-(i+s)}\varphi(2^{i+s}x)$ approaches zero as $s\to\infty$. Therefore, $\{4^{-s}g_{\mathrm{e}}(2^s x)\}_{s=0}^{\infty}$ is a Cauchy sequence in the Banach space Y. We can now define a function $T_{\mathrm{e}}: X\to Y$ by

$$T_{\mathrm{e}}(x)=\lim_{s\to\infty}4^{-s}g_{\mathrm{e}}(2^s x)=\lim_{s\to\infty}4^{-s}f_{\mathrm{e}}(2^s x)$$

for all $x\in X$. Thus,

$$\|g_{\mathrm{e}}(x)-T_{\mathrm{e}}(x)\|\leqslant\frac{1}{8}\sum_{i=0}^{\infty}4^{-i}\varphi(2^i x).$$

From the definition of $g_{\mathrm{e}}(x)$ and (15), one finds that

$$\begin{aligned}\|f_{\mathrm{e}}(x)-T_{\mathrm{e}}(x)\| &= \left\|g_{\mathrm{e}}(x)-\frac{A}{6}f(0)-T_{\mathrm{e}}(x)\right\| \\ &\leqslant \|g_{\mathrm{e}}(x)-T_{\mathrm{e}}(x)\|+\left\|\frac{A}{6}f(0)\right\| \\ &\leqslant \frac{1}{8}\sum_{i=0}^{\infty}4^{-i}\varphi(2^i x)+\frac{\varphi(0)}{6}.\end{aligned}$$

In a similar manner, we can prove that $\{2^{-s}f_{\mathrm{o}}(2^s x)\}_{s=0}^{\infty}$ converges in the Banach space Y. Define a function $T_{\mathrm{o}}: X\to Y$ by

$$T_{\mathrm{o}}(x)=\lim_{s\to\infty}2^{-s}f_{\mathrm{o}}(2^s x)$$

for all $x\in X$. Thus,

$$\|f_{\mathrm{o}}(x)-T_{\mathrm{o}}(x)\|\leqslant\frac{1}{4}\sum_{i=0}^{\infty}2^{-i}\varphi(2^i x).$$

Define a function $T: X\to Y$ by

$$T(x)=T_{\mathrm{e}}(x)+T_{\mathrm{o}}(x)$$

for all $x\in X$. Then we obtain

$$\begin{aligned}\|f(x)-T(x)\| &\leqslant \|f_{\mathrm{e}}(x)-T_{\mathrm{e}}(x)\|+\|f_{\mathrm{o}}(x)-T_{\mathrm{o}}(x)\| \\ &\leqslant \frac{1}{8}\sum_{i=0}^{\infty}4^{-i}\varphi(2^i x)+\frac{1}{4}\sum_{i=0}^{\infty}2^{-i}\varphi(2^i x)+\frac{\varphi(0)}{6}\end{aligned} \tag{21}$$

for all $x\in X$. Next, we will show that T satisfies (3). Define the even part and the odd part of Df by $Df_{\mathrm{e}}(x_1,\dots,x_n)=\frac{1}{2}(Df_{[1]}+Df_{[-1]})$ and $Df_{\mathrm{o}}(x_1,\dots,x_n)=\frac{1}{2}(Df_{[1]}-Df_{[-1]})$ where $f_{[q]}\equiv f(qx_1,\dots,qx_n)$. Then for a positive integer s and for all $x_1,x_2,\dots,x_n\in X$, we get

$$\begin{aligned}\|Df_{\mathrm{e}}(2^s x_1,\dots,2^s x_n)\| &= \tfrac{1}{2}\left\|Df_{[2^s]}\right\|+\tfrac{1}{2}\left\|Df_{[-2^s]}\right\| \\ &\leqslant \phi(2^s x_1,2^s x_2\dots,2^s x_n).\end{aligned}$$

Dividing the above inequality by 4^s and taking the limit as $s\to\infty$, we then have $DT_{\mathrm{e}}(x_1,x_2,\dots,x_n)=0$ for all $x_1,x_2,\dots,x_n\in X$. Similarly, we can show that $DT_{\mathrm{o}}(x_1,x_2,\dots,x_n)=0$. Hence $T=T_{\mathrm{e}}+T_{\mathrm{o}}$ satisfies (3).

Next we will prove that T is unique. Suppose that there exists another function $T': X\to Y$ such that T' satisfies (3) and (14). Since we already proved in Theorem 1 that T_{e} and T_{o} satisfy the quadratic and the additive functional equations, respectively, we have that $T_{\mathrm{e}}(rx)=r^2T_{\mathrm{e}}(x)$ and $T_{\mathrm{o}}(rx)=rT_{\mathrm{o}}(x)$ for every $r\in\mathbb{Q}$ and for every $x\in X$ (see pp. 35–6, 100–1 in Ref. 1). By the definition of T and the triangle inequality, one gets that $\|T(x)-T'(x)\|\leqslant\|T_{\mathrm{e}}(x)-T'_{\mathrm{e}}(x)\|+\|T_{\mathrm{o}}(x)-T'_{\mathrm{o}}(x)\|$. For any positive integer s and for each $x\in X$,

$$\begin{aligned}\|T_{\mathrm{e}}(x)-T'_{\mathrm{e}}(x)\| \\ &= 4^{-s}\|T_{\mathrm{e}}(2^s x)-T'_{\mathrm{e}}(2^s x)\| \\ &\leqslant 4^{-s}\|f_{\mathrm{e}}(2^s x)-T_{\mathrm{e}}(2^s x)\| \\ &\quad+4^{-s}\|f_{\mathrm{e}}(2^s x)-T'_{\mathrm{e}}(2^s x)\| \\ &\leqslant \frac{1}{4}\sum_{i=0}^{\infty}4^{-(i+s)}\varphi(2^{i+s}x)+4^{-s}\frac{\varphi(0)}{3}.\end{aligned}$$

We now take the limit as $s\to\infty$. Since $\sum_{i=0}^{\infty}4^{-i}\varphi(2^i x)$ converges, $\sum_{i=0}^{\infty}4^{-(i+s)}\varphi(2^{i+s}x)$ tends to zero as $s\to\infty$. Then we can conclude that $T_{\mathrm{e}}(x)=T'_{\mathrm{e}}(x)$ for all $x\in X$. Similarly, $T_{\mathrm{o}}(x)$ and $T'_{\mathrm{o}}(x)$ are proved to be equal for all $x\in X$. Hence $T(x)=T'(x)$ for all $x\in X$.

For the case when condition (ii) holds, the proof is similar. Condition (ii) implies that $\varphi(0)=0$. Thus $f(0)=0$. Starting by setting $(x_1,x_2,\dots,x_n)=(\frac{x}{2},\frac{x}{2},0,\dots,0)$ in (13),

$$\left\|f(x)-3f\left(\frac{x}{2}\right)-f\left(\frac{-x}{2}\right)\right\|\leqslant\frac{\varphi(2^{-1}x)}{2}.$$

It follows from the definitions of f_{e} and f_{o} that

$$\left\|f_{\mathrm{e}}(x)-4f_{\mathrm{e}}\left(\frac{x}{2}\right)\right\|\leqslant\frac{\varphi(2^{-1}x)}{2}$$

and

$$\left\|f_{\mathrm{o}}(x)-2f_{\mathrm{o}}\left(\frac{x}{2}\right)\right\|\leqslant\frac{\varphi(2^{-1}x)}{2}.$$

The previous two inequalities can be extended to

$$\left\|f_{\mathrm{e}}(x)-4^{s}f_{\mathrm{e}}(2^{-s}x)\right\|\leqslant\frac{1}{8}\sum_{i=1}^{s}4^{i}\varphi(2^{-i}x)$$

and

$$\left\|f_{\mathrm{o}}(x)-2^{s}f_{\mathrm{o}}(2^{-s}x)\right\|\leqslant\frac{1}{4}\sum_{i=1}^{s}2^{i}\varphi(2^{-i}x)$$

for a positive integer s and for all $x\in X$. The rest of the proof can be carried out in a similar fashion. □

Corollary 1 *If a function* $f : X \to Y$ *satisfies* $\|Df(x_1,x_2,\dots,x_n)\|\leqslant\varepsilon$, *for all* $x_1,x_2,\dots,x_n\in X$ *for some* $\varepsilon>0$, *then there exists a unique function* $T : X\to Y$ *that satisfies* (3) *and, for all* $x\in X$, $\|f(x)-T(x)\|\leqslant\frac{5}{6}\varepsilon$.

Proof: From Theorem 2, let $\phi(x_1,x_2,\dots,x_n)=\varepsilon$ for all $x_1,x_2,\dots,x_n\in X$. Consequently, $\varphi(x)=\varepsilon$ for all $x\in X$. Being in accordance with condition (i), it follows from the theorem that there exists a unique function $T : X\to Y$ such that

$$\begin{aligned}\|f(x)-T(x)\|&\leqslant\|f_{\mathrm{e}}(x)-T_{\mathrm{e}}(x)\|+\|f_{\mathrm{o}}(x)-T_{\mathrm{o}}(x)\|\\&\leqslant\frac{1}{8}\sum_{i=0}^{\infty}4^{-i}\varepsilon+\frac{1}{4}\sum_{i=0}^{\infty}2^{-i}\varepsilon+\frac{\varepsilon}{6}=\frac{5}{6}\varepsilon\end{aligned}$$

for all $x\in X$. □

Corollary 2 *If a function* $f : X\to Y$ *satisfies*

$$\|Df(x_1,\dots,x_n)\|\leqslant\varepsilon\sum_{i=1}^{n}\|x_i\|^{p}$$

for all $x_1,\dots,x_n\in X$ *for some* $\varepsilon>0$ *where* $p\in(0,1)\cup(2,\infty)$, *then there exists a unique function* $T : X\to Y$ *that satisfies* (3) *and*

$$\|f(x)-T(x)\|\leqslant\frac{2\varepsilon\,|3-2^{p}|}{|4-2^{p}|\,|2-2^{p}|}\,\|x\|^{p}$$

for all $x\in X$.

Proof: According to Theorem 2, let $\phi(x_1,\dots,x_n)=\varepsilon\sum_{i=1}^{n}\|x_i\|^{p}$ for all $x_1,x_2,\dots,x_n\in X$. Then we obtain that $\varphi(x)=2\varepsilon\|x\|^{p}$ for all $x\in X$ and $\varphi(0)=0$. If $p<1$, then condition (i) holds. Applying Theorem 2, we get that

$$\begin{aligned}\|f(x)-T(x)\|&\leqslant\frac{1}{8}\sum_{i=0}^{\infty}4^{-i}\cdot2\varepsilon\left\|2^{i}x\right\|^{p}\\&\quad+\frac{1}{4}\sum_{i=0}^{\infty}2^{-i}\cdot2\varepsilon\left\|2^{i}x\right\|^{p}\\&=\frac{\varepsilon\|x\|^{p}}{4-2^{p}}+\frac{\varepsilon\|x\|^{p}}{2-2^{p}}\\&=\frac{2\varepsilon(3-2^{p})}{(4-2^{p})(2-2^{p})}\|x\|^{p}\end{aligned}$$

for all $x\in X$. If $p>2$, then condition (ii) holds and we obtain a similar result. □

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