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The Existence and Stability of the Travelling Wave Solution of a Gompertz Growth with the Simplest Nonlinear Advection Model

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ABSTRACT: We show the existence of a travelling wave solution for a simplest nonlinear advection and the Gompertz reaction model. We prove that the solution is perturbatively stable without any restriction on the parameters of the Gompertz model but the stability of one of its trivial solutions is subject to a restriction on one of the values of the Gompertz reaction parameters. We also show that the solution is Poincare stable around one of it critical points provided the wave velocity of the traveling wave solution is greater than the product of the two reaction parameters.

KEYWORDS: Gompertz reaction model, Non-linear advection model, partial differential equation (PDE), traveling wave solution of a PDE, perturbative stability, Poincare stability.

A simplest nonlinear reaction-advection model is of the form

$$\frac{\partial u}{\partial t} - ku\frac{\partial u}{\partial x} = f \circ u \tag{1.1}$$

where f is a non-linear function which represents the growth rate or reaction rate of the process, with the advection term $-ku\frac{\partial u}{\partial x}$ in which $(-ku^2/2)$ is known as the flux of the system. In general, equation (1.1) has also been referred to as "a growth model with advection-reaction". This model, or even a general reaction-advection model (also known as reaction-convection model),

$$\frac{\partial u}{\partial t} + \frac{\partial F \circ u}{\partial x} = f \circ u \tag{1.2}$$

has been studied generally by Murray since late 1960's and reviewed in his two books, Murray [1,2]. In those books, Murray also has shown that equation (1.1) with $k=1$ can be considered as a limiting case of a nonlinear reaction-advection-diffusion model

$$\frac{\partial u}{\partial t} - \gamma u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} + R \circ u \tag{1.3}$$

By rescaling x such that $x = \gamma y$, equation (1.3) becomes

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial y} = (1/\gamma^2)\frac{\partial^2 u}{\partial y^2} + R \circ u \tag{1.4}$$

and by choosing γ as a sufficiently large number so that the diffusion term can be neglected, equation (1.4) tends to equation (1.1) with k=1. It is interesting to note that the term "nonlinear advection-reaction equation" is also popular, but is still not standardized. For example, to many, the term "nonlinear" is referring to the reaction term only, and thus to them the "nonlinear advection-reaction equation" or "reaction-advection equation" means

$$\frac{\partial u}{\partial t} + v(x,t)\frac{\partial u}{\partial x} = f \circ u \tag{1.5}$$

In fact many papers on the numerical aspect of the "nonlinear reaction-advection equation" or "nonlinear reaction-advection" deal with equation (1.5) with a nonlinear function f, and there are papers with a title "reaction-advection equation" which refers to equations (1.1) or (1.3) with a linear function f or R. Examples of two well known recent papers on the computational issues of equation (1.3), in which R is linear, are by Kojouharov and Welfert [3], and Mickens [4]. However, we are not interested in any numerical aspect of these equations.

Some special cases of reaction or growth model f in equation (1.1) have been considered to show the existence of its solution perhaps with some initial conditions or boundary conditions. For example, Tarter [5] has discussed the existence of a periodic solution for a general nonlinear partial differential equation (PDE) which includes equation (1.1) as its special case; Toro and Titarev [6] discussed the existence of the generalized Riemann problem for equation (1.1);

while Lika and Hallam [7] and Khadizah and Shaharir [8] have not only discussed the existence of a travelling wave solution of equation (1.1) for the case of f given by the well known logistic curve

$$f(w) = rw(b\text{-}w),\ w = u(x,t)\ ,\ r{>}0,\ b{>}0 \qquad (1.6)$$

but also its stability. The purpose of this paper is to report our study [9] on the nature of the solution of equation (1.1) for the case of a Gompertz growth or reaction model, namely

$$f(w) = rw\, Ln(b/w),\ r>0, b>0\ , w>0 \qquad (1.7)$$

The Gompertz growth model without advection, namely equation (1.1) with $k = 0$, was first introduced by Gompertz in the early 19th century, but interest in it has not subsided ever since. In fact even since five years ago, based on "Ingenta Select" search engine, there are more than 340 papers referring to such models in various studies and more than 20 of them are with the titles which specifically refer to Gompertz growth or model such as in demography by Yi et al. [10] and Haybittle [11]; cell growth study by Rossi et al. [12] and Tomas et al. [13]; and a study of drug disposition by Easton [14,15]. But none of those papers are concerning the Gompertz growth model with a nonlinear advection term, namely equations (1.1) and (1.7). Many refer equation (1.1) as a simplest nonlinear advection with the growth (or reaction) model f, and hence in the case f given by equation (1.7), it is accordingly refer to as a simplest nonlinear advection with the Gompertz reaction or Gompertz growth model. We prefer to refer it as the Gompertz growth model with a simplest nonlinear advection.

Following the method which has been successfully applied by Lika and Hallam [7] and improved by Khadizah and Shaharir [8], we let

$u(x,t) = W(x{+}ct)$,

c is a non-zero constant of real number, (2.1)

then W naturally satisfies

$(c{-}kW)W' {=} rW\, Ln_{\circ}(b/W)\ , c\text{-}kW{\neq}0, W{>}0$ (2.2)

where the prime denotes the derivative with respect to

$z = x + ct.$ (2.3)

Using the method of separation of variables and the well known special function Li (natural logarithm-integral, see for example, Gradshleyn and Ryzhik [16]),

$$Li(\xi) = \int_a^{\xi} \frac{1}{Ln(y)}\,dy\ , a{>}0 \qquad (2.4)$$

equation (2.2) gives, for the case $w{\neq}0{+}, b$

$kb\, Li\,(w/b) - c\, Ln^{1/2}(Ln\,(w/b))^{1/2} = rz + Q$, Q is a constant,

$w = W(z)$ (2.5)

The existence of the implicit solution in equation (2.4) can be shown by using the well known implicit function theorem, which requires $\frac{\partial F}{\partial w} \neq 0$ in some region of the domain of F,

$F(w, z){=} kb\, Li\,(w/b) - c\, Ln\,|\,(Ln\,(w/b))\,|\, \text{-}\, rz - Q\ , w = W(z)$ (2.6)

But

$$\frac{\partial F}{\partial w} = \frac{1}{w\ \ Ln(w/b)}[\ kw{-}c\ ]{\neq}0 \text{ for all } (w,z)\ , kw\text{-}c{\neq}0 \qquad (2.7)$$

which shows the existence of $w = q(z)$, or equivalently the existence of $W{\neq}0{+}, b$ and hence $u{\neq}0{+}, b$ implicitly in equation (2.4) in the region $\{(w,z){:}\ kw - c \neq 0\}$. But $W = 0+$ and b are also solutions for equation (2.2), by insisting that $W'(0+) = W'(b) = 0$.

The above analysis shows that we have proved the following theorem:

Theorem 1

Other than $W = 0{+}, b$, for every $c > 0$, there exists a travelling wave solution $u(x,t) = W(x{+}ct)$ for equations (1.1) and (1.4), $0 < W < b$, $W \neq c/k$.

Now following Lika and Hallam [7] and our paper Khadizah and Shaharir [8], we study the perturbative stability of the traveling waves (1.1) and (1.4) given in the above theorem. For this, let

$$u(x,t) = W(z) + v(z,t)\ , z = x + ct \qquad (3.1)$$

where W is a known solution given in the above theorem and v is a small perturbation term (with respect to a norm) as such $v(z,s) = 0$ for z in an interval $(-\infty, a)$ for some a, and nonlinear terms of v can be neglected. Then naturally v satisfies, after using also equation (2.2),

$$\frac{\partial v}{\partial t} + [c - kW]\frac{\partial v}{\partial z} + [2q \circ W - r - kW']\quad v \neq 0 \qquad (3.2a)$$

$2q(w) = Ln(w/b) + 2r\ , w = W(z) > 0$ (3.2b)

Thus,

$v(z,t) = \exp(\text{-}\lambda t)\, g(z)$ (3.3)

where g satisfies an eigen-value or an eigen-vector problem,

$(c\text{-}kW(z))g'(z){+}[2q(W(z))\text{-}r\text{-}kW'(z)]g(z){=}\lambda g(z),\ W{>}0$ (3.4)

in which g is needed to be bounded in $[a,\infty)$ for some positive values of λ for the stability of the solution concerned.

Lika and Hallam [7] have proved an important Lemma in which the finiteness of the integral of the form

$$\left(-\int_a^z \left(\frac{2rW(y) - rb - \lambda}{c - kW(y)}\right) dy\right),\ z \text{ in } [-L,\infty),$$

where $(c{-}kW)W' = rW(b\text{-}W)$ is needed. To achieve this they have proposed some unnecessary assumptions and provided a long and winding arguments. Thus we have improved the method of proving the finiteness of the integral without the restrictive assumptions (also stated in Khadizah and Shaharir[8]), and the same method is applied to the present problem arises in this paper. We obtain the following lemma.

LEMMA

The solution g of equation (3.4), $c>0$, $g(a)\neq 0$, a real, is bounded in $[a,\infty)$ for some positive λ for arbitrary $b>0$ in the case of $W>0$. In the case of $W=0+$, the solution g satisfies the same property, provided $0<b<1$.

Proof

Following the strategy of Lika and Hallam[7], let $h=(c-kW)g$, since $c-kW$ is not zero. Then the equation in the lemma (i.e equation (3.4)) becomes

$h' + F h = 0$, $F(z) = [2q(W(z)) - r - \lambda]/(c-kW)$

Thus

$$h(z) = h(a)\,exp\left(-\int_a^z \frac{2q(W(y))-r-\lambda}{\left(c-kW(y)\right)}dy\right)$$

Therefore

$$g(z) = \frac{c-kW(a)}{c-kW(z)} g_a exp\left(-\int_a^z \frac{2q(W(y))-r-\lambda}{\left(c-kW(y)\right)}dy\right), g_a = g(a) \tag{L1}$$

We have to show that this g is bounded so that v in equation (3.3) is bounded for $c>0$ and $\lambda>0$.

First , take $a=-L<0$ and $W(0)=b/2$.

For every $c>0$ and a specific $L>0$, let $\lambda = 2W(L) - b/2$. Then there exists a positive constant J such that $|g(z)|<J$ for every z in the interval $[-L,\infty)$. Since W is monotonically increasing, then we can conclude that

$$0<W(-L)<b/2<W(L)<b. \tag{L2}$$

Now we have

$$|g(z)| = |g_a|\left|\frac{c-kW(a)}{c-kW(z)}\right| exp\left(-\int_{-L}^z \frac{2q(W(y))-r-\lambda}{\left(c-kW(y)\right)}dy\right),$$

$$< |g_a|\left|\frac{c-kb/2}{c-kb}\right| exp\left(-\int_{-L}^z \frac{2q(W(y))-r-\lambda}{\left(c-kW(y)\right)}dy\right),$$

where equation (L2) has been used and the fact that $0<W<b$

$$< |g_a|\left|\frac{2c-kb}{c-kb}\right| exp\left(-\int_{-L}^z \frac{2q(W(y))-r-\lambda}{\left(c-kW(y)\right)}dy\right). \tag{L3}$$

Thus, g is bounded if

$$N(z) = \left(-\int_{-L}^z \frac{2q(W(y))-r-\lambda}{\left(c-kW(y)\right)}dy\right) \text{ is bounded in } [-L,\infty). \tag{L4}$$

Now, notice that

$$\left(-\int_{-L}^z\left(\frac{2rW(y)-r-\lambda}{c-kW(y)}\right)dy\right) = \int_{-L}^z \frac{rLn(W(y)/b)+r-\lambda}{\left(c-kW(y)\right)}dy,$$

$$= \int_{-L}^z\left[\frac{-W'(y)}{W(y))} + \frac{r-\lambda}{(c-kW(y))}\right]dy, \; W>0,\, c\text{-}kW\neq 0,\, W\neq 0+ \tag{L5}$$

after using equation (2.2).

But

$$\int_{-L}^z\left[\frac{W'(y)}{W(y)}\right]dy = -Ln\left|\frac{W(z)}{W(-L)}\right| < \infty, \; 0<W<b,\, W\neq 0+ \tag{L6}$$

and

$$\int_{-L}^z\left[\frac{r-\lambda}{(c-kW(y))}\right]dy = \int_{-L}^z\left[\frac{(r-\lambda)W'(y)}{W(y)Ln(W(y)/b)}\right]dy, \text{ by equation (2.2)}$$

$$= (r-\lambda)\,[\,Ln(Ln|W(z)/W(-L)| < \infty \tag{L7}$$

By equations (L5)-(L7), we have proved that N in equation (L4) is bounded for all z, and hence g in equation (L3) is bounded as asserted in the Lemma, for the case $W\neq b$, $0+$.

In the case of $W=b$, we need to consider (since in this case $W'=0$), from equation (3.2),

$(c-kb)g' + rg = \lambda g$

which shows that g is bounded provided

$(\lambda-r)/(c-kb)\leq 0$

or equivalently,

$0<\lambda<r$, $c>kb$; or $\lambda>r$, $c<kb$; or $\lambda=r$.

Thus the Lemma is also true for the case $W=b$.

In the case of $W=0+$, we need to consider the perturbative term v which in this case satisfies

$$\frac{\partial v}{\partial t} + (c-kv)\frac{\partial v}{\partial z} = rvLn(b/v)$$

and hence its linearised form is

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial z} - rLn(b)v = 0 \tag{L8}$$

Now let $v(z,t) = \exp(-\lambda t)g(z)$ and hence g satisfies $cg' - rLn(b)\,g = \lambda g$ which shows that g is bounded provided $0<\lambda\leq -r\,Ln(b)$ since $c>0$ provided $0<b<1$, since $r>0$. This completes the proof of the Lemma .

Using the above Lemma, we have proved the following theorem.

THEOREM 2

The non-zero travelling wave solution of equations (1.1) and (1.7) is perturbatively stable. It zero solution, namely W=0+ , is perturbatively stable provided 0<b<1. The other trivial solution, W=b, is perturbatively stable without any restriction.

The following is the study of the nature of the solution of equations (1.1) and (1.7) at its critical points, i.e at the critical points of equation (2.2). It is well known (Hartman's theorem) that for any ordinary differential equation (ODE) $y' = F(y)$ provided F is smooth enough (e.g satisfies a the Lipschitz condition), the behaviour of y is possibly not topologically a straight line only in the neigbourhood of the critical points of F. This is not to be confused with the behaviour of the trivial solutions of y namely $y = c$ where c, $F(c) = 0$, is a critical element of the ODE.

According to the well known theory of the Poincare methods for understanding of the nature of a critical point of equation (2.2), we need to linearise

$$F(y) = [\, r\, y\, Ln\,(b/y)]/(c\text{-}ky), y>0 \qquad (4.1)$$

at $y=b$ or possibly y = 0+ respectively.

At a critical point $y = y_c$ of F, the linearised part of it is given by $A_F(y_c, y) = F(y_c) + F'(y_c)(y - y_c)$ where $F'(y) = r[Ln(b/y) - 1]/(c\text{-}ky) + rky^2\, Ln(b/y)]/(c\text{-}ky)^2$, $y \neq 0+, y>0.$ (4.2)

Accordingly, for studying the qualitative nature of the solution for equation (2.2) at $W = b$, we must solve

$$W' = [r/(kb\text{-}c\,)\,]\,(W\text{-}b) \qquad (4.3)$$

and hence

$$W(z) = b + W_0 \exp[rz/(kb\text{-}c)] \qquad (4.4)$$

which shows that W is bounded provided $r/(kb\text{-}c) < 0$, i.e. $c > kb$, since r>0. Equation (4.4) is the phase potrait for W.

At W = 0+, we need to show first that $F(0+) = 0$. It is sufficient to show that $y\, Ln(y)$ tends to 0 as y tends to 0+ or equivalently $(1/y)\, Ln(1/y)$ tends to 0 as y tend to ∞. It is sufficient to show that $Ln(y)/y$ tends to 0 as y tends to ∞. But by L'Hopital's rule, it is indeed true that $Ln(y)/y$ tends to 0 as y tends to ∞. Thus, we may assume that W = 0+ is a critical point of equation (2.2). Next we have to linearise F in equation (4.1) at y = 0+. For this, we have to determine $F'(0+)$ from the definition of the derivative. We have

$F'(0+) =$

$$\lim_{h\to 0}\frac{F(0^+ + h) - F(0^+)}{h} = \lim_{h\to 0}\frac{F(h+)}{h} = (r/c)\lim_{h\to 0} Ln(b/h+) = \infty$$

for any b>0, which shows that W near its critical point W = 0+ cannot be linearised, and hence the Poincare stability method is not applicable at that point.

We conclude that $W(z)=u(x,t)$, $z=x+ct$, is stable near its critical points $W = b$ provided $c>kb$ where c is the wave velocity of the traveling wave solution u of equations (1.1) and (1.7); otherwise it is unstable. Based on the theory of Poincare stability, we can say that the travelling wave solution of equations (1.1) and (1.7) with the condition on its wave velocity given above finally stays around at the vicinity of its critical points b.

References

1. Murray JD (1994) *Mathematical Biology,* 3rd ed , Ch. 11. Springer, Berlin.
2. Murray JD (1977) *Lectures on Nonlinear-Differential-Equation Models in Biology*. Ch.5. Clarendon Press.
3. Kojouharov HV and Welfert BD (2001) A new numerical approach for the solution of nonlinear advection-reaction equation. *Jour of Applied Sc and Computation* **8(2)**, 119-26.
4. Mickens RE (2003) Exact solution to finite difference of a nonlinear reaction-advection equation. Implications for numerical analysis. *J Difference Eqns and Its Applications* **9(11)**, 995-1006
5. Tartar L (1989) Oscillations in nonlinear partial differential equations: compensated compactness and homogenization. In: *Nonlinear Systems of Partial Differential Equations in Applied Mathematics* (Edited by Nicolaenko B, Holm DD and Hyman JM), pp 243-66. American Math Soc, USA.
6. Toro TF, Titarev VA (2002) Solution of the generalized Riemann problem for advection-reaction equations. *Proc Math Phys and Engineering Sc* (*The Royal Soc*) 458 (2018), 271-81.
7. Lika K and Hallam TG (1999) Travelling wave solution of a non-linear reaction-advection equation. *J Math Biology* 38, 346-58.
8. Khadizah BtG and Shaharir bMZ (2007) Kestabilan usikan dan Poincare bagi penyelesaian gelombang kembara untuk pertumbuhan lojistik dengan model adveksi tak linear. *Matematika*. Number 1 Vol 23, June 2007 (In Malay).
9. Khadizah BtG (2004) Kajian Kestabilan Tindakbalas-Adveksi dengan Model Pertumbuhan Gompertz. *Masters Thesis* at Pusat Pengajian Sains Matematik, Universiti Kebangsaan Malaysia, Malaysia. (in Malay). Unpublished. (In Malay).
10. Yi Z, Zhenglian W, Zhongdong M and Chunjun C (2000) A simple method for projecting or estimating alpha and beta: an extension of the Brass relational Gompertz fertility model. *Population Res and Policy Rev* 19(6), 525-49.
11. Haybittle J (1998) The use of Gompertz function to relate changes in life expectancy to standardized mortality rate. *Int J Epidemiology* 27(5), 885-9.
12. Rossi S, Deslauriers A and Morin H (2003) Application of the Gompertz equations for the study of xylem cell development. *Dendrochronologia* 2(1), 33-5.
13. Tomas MSJ, Labanda EBD, de Ruiz Holgado AP and Nader-Macias ME (2002) Estimation of vaginal probiotic lactobacilli growth parameters with the application of the Gompertz model. *Canadian J Microbiology* 48(1), 82-92.
14. Easton DM (2002) Gompertz pharmacokinetic model for drug disposition. *Pharmaceutical Res* 19(4), 463-9.
15. Easton DM (1999) X-ray survival as Gompertz Growth in number killed. *J. Theor. Biology* 196(1), 1-8.
16. Gradshteyn IS and Ryzhik IM (1965) *Table of Integrals Series and Products*. Academic Press, N.York.