On Quasi-gamma-ideals in Gamma-semigroups

Ronnason Chinram*

Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, 90112, Thailand.

* Corresponding author, E-mail: ronnason.c@psu.ac.th

Received 18 Jan 2006
Accepted 10 May 2006

ABSTRACT: The concept of quasi-ideals in semigroups was introduced in 1956 by O. Steinfeld. The class of quasi-ideals in semigroups is a generalization of one-sided ideals in semigroups. It is well-known that the intersection of a left ideal and a right ideal of a semigroup \( S \) is a quasi-ideal of \( S \) and every quasi-ideal of \( S \) can be obtained in this way. In 1981, M. K. Sen have introduced the concept of \( \Gamma \)-semigroups. One can see that \( \Gamma \)-semigroups are a generalization of semigroups. In this research, quasi-\( \Gamma \)-ideals in \( \Gamma \)-semigroups are introduced and some properties of quasi-\( \Gamma \)-ideals in \( \Gamma \)-semigroups are provided.

KEYWORDS: \( \Gamma \)-semigroups, quasi-\( \Gamma \)-ideals, minimal quasi-\( \Gamma \)-ideals, quasi-simple \( \Gamma \)-semigroups.

INTRODUCTION

Let \( S \) be a semigroup. A nonempty subset \( Q \) of \( S \) is called a quasi-ideal of \( S \) if \( Q \cap QS \subseteq Q \) and \( Q \cap SQ \subseteq Q \). Then \( Q \) is a quasi-ideal of \( S \). The concept of quasi-ideals in semigroups was introduced in 1956 by O. Steinfeld (see [1]). The author has studied some properties of quasi-ideals in semigroups (See [2] and [3]).

Example 1.1. Let \( S = \{0, 1\} \). Then \( S \) is a semigroup under usual multiplication. Let \( Q = \{0, 1/2\} \). Thus \( SQ \cap QS \subseteq Q \). Therefore, \( Q \) is a quasi-ideal of \( S \).

A nonempty subset \( L \) of \( S \) is called a left ideal of \( S \) if \( SL \subseteq L \) and a nonempty subset \( R \) of \( S \) is called a right ideal of \( S \) if \( RS \subseteq R \). Clearly, every left ideal and every right ideal of a semigroup is a subsemigroup of \( S \). Next, let \( L \) and \( R \) be a left ideal and a right ideal of a semigroup \( S \). By the definition of quasi-ideals of semigroups, it is easy to prove that \( L \cap R \) is a quasi-ideal of \( S \) (See [4]).

Example 1.2. Let \( Z \) be the set of all integers and \( M_2(Z) \), the set of all \( 2 \times 2 \) matrices over \( Z \). We have known that \( M_2(Z) \) is a semigroup under the usual multiplication. Let

\[
L = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in Z \right\}
\]

and

\[
R = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in Z \right\}.
\]

Then \( L \) is a left ideal of \( M_2(Z) \), \( R \) is a right ideal of \( M_2(Z) \) and

\[
L \cap R = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x \in Z \right\}
\]

is a quasi-ideal of \( M_2(Z) \).

In 1981, the notion of \( \Gamma \)-semigroups was introduced by M. K. Sen (See [5], [6], and [7]). Let \( M \) and \( \Gamma \) be any two nonempty sets. If there exists a mapping \( M \times \Gamma \times M \rightarrow M \), written \( (a, \gamma, b) \) by \( a\gamma b \), \( M \) is called a \( \Gamma \)-semigroup if \( M \) satisfies the identities \( a\gamma(b\mu) = (a\gamma b)\mu \) for all \( a, b, c \in M \) and \( \gamma, \mu \in \Gamma \). Let \( K \) be a nonempty subset of \( M \). Then \( K \) is called a sub \( \Gamma \)-semigroup of \( M \) if \( a\gamma b \in K \) for all \( a, b \in K \) and \( \gamma, \mu \in \Gamma \).

Example 1.3. Let \( S \) be a semigroup and \( \Gamma \) be any nonempty set. Define a mapping \( S \times \Gamma \times S \rightarrow S \) by \( a\gamma b \in K \) for all \( a, b \in S \) and \( \gamma, \mu \in \Gamma \). Then \( S \) is a \( \Gamma \)-semigroup.

Example 1.4. Let \( M = \{0, 1\} \) and

\[
\Gamma = \left\{ \frac{1}{n} \mid n \text{ is a positive integer} \right\}.
\]

Then \( M \) is a \( \Gamma \)-semigroup under the usual multiplication. Next, let \( K = \{0, 1/2\} \). We have that \( K \) is a nonempty subset of \( M \) and \( a\gamma b \in K \) for all \( a, b \in K \) and \( \gamma, \mu \in \Gamma \). Then \( K \) is a sub \( \Gamma \)-semigroup of \( M \).

From example 1.3, we have that every semigroup is a \( \Gamma \)-semigroup. Therefore, \( \Gamma \)-semigroups are a generalization of semigroups.

In this research, we generalize some properties of quasi-ideals of semigroups to some properties of quasi-\( \Gamma \)-ideals in \( \Gamma \)-semigroups.
Main Results

Let $M$ be a $\Gamma$-semigroup. A nonempty subset $Q$ of $M$ is called a quasi-$\Gamma$-ideal of $M$ if $\cap \{ MG \cap \Gamma M \subseteq Q \}$. Let $Q$ be a quasi-$\Gamma$-ideal of $M$. Then $QGM \subseteq MG \cap \Gamma M \subseteq Q$. This implies that $Q$ is a sub-$\Gamma$-semigroup of $M$.

Example 2.1. Let $S$ be a semigroup and $\Gamma$ be any nonempty set. Define a mapping $\delta : S \times \Gamma \times S \to S$ by $a \delta b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. From example 1.3, $S$ is a $\Gamma$-semigroup. Let $Q$ be a quasi-$\Gamma$-ideal of $S$. Thus $SQ \cap QS \subseteq Q$. We have that $S^G \cap QS = SQ \cap QS \subseteq Q$. Hence, $Q$ is a quasi-$\Gamma$-ideal of $S$.

Example 2.1 implies that the class of quasi-$\Gamma$-ideals in $\Gamma$-semigroups is a Generalization of quasi-ideals in semigroups.

Theorem 2.1. Let $M$ be a $\Gamma$-semigroup and $Q_i$ a quasi-$\Gamma$-ideal of $M$ for each $i \in I$. If $\bigcap_{i \in I} Q_i$ is a nonempty set, then $\bigcap_{i \in I} Q_i$ is a quasi-$\Gamma$-ideal of $M$.

Proof. Let $M$ be a $\Gamma$-semigroup and $Q_i$ a quasi-$\Gamma$-ideal of $M$ for each $i \in I$. Assume that $\bigcap_{i \in I} Q_i$ is a nonempty set. Take any $a, b \in \bigcap_{i \in I} Q_i$, $m_i \in M$ and $\gamma, \mu \in \Gamma$ such that $m, \mu b = a \gamma m_i$. Then $a, b \in Q_i$, for all $i \in I$. Since $Q_i$ is a quasi-$\Gamma$-ideal of $M$ for all $i \in I$, $m, \mu b = a \gamma m_i \in \cap_{i \in I} \Gamma M \subseteq Q_i$ for all $i \in I$. Therefore $m, \mu b = a \gamma m_i \in \bigcap_{i \in I} Q_i$. Thus $M \bigcap_{i \in I} Q_i \cap \Gamma M \subseteq \bigcap_{i \in I} Q_i \cap \Gamma M \subseteq \bigcap_{i \in I} Q_i \Gamma M \subseteq Q_i$. Hence, $\bigcap_{i \in I} Q_i$ is a quasi-$\Gamma$-ideal of $M$.

In Theorem 2.1, the condition $\bigcap_{i \in I} Q_i$ is a nonempty set is necessary. For example, let $N$ be the set of all positive integers and $\Gamma = \{ 1 \}$. Then $\bigcap_{n \in N} Q_n$ is a $\Gamma$-semigroup. For $n \in N$, let $Q_n = \{ n + 1, n + 2, n + 3, \ldots \}$. It is easy to show that each $Q_n$ is a quasi-$\Gamma$-ideal of $M$ for all $n \in N$ but $\bigcap_{n \in N} Q_n$ is an empty set.

Let $A$ be a nonempty subset of a $\Gamma$-semigroup $M$ and $\mathcal{A} = \{ Q \mid Q$ is a quasi-$\Gamma$-ideal of $M$ containing $A$ $\}$. Then $\mathcal{A}$ is a nonempty set because $M \in \mathcal{A}$. Let $\{ A \}$ be $\bigcap_{Q \in \mathcal{A}} Q$. $\{ A \}$ is a quasi-$\Gamma$-ideal of $M$. Moreover, $\{ A \}$ is the smallest quasi-$\Gamma$-ideal of $M$ containing $A$. $\{ A \}$ is called the quasi-$\Gamma$-ideal of $M$ Generated by $A$.

Theorem 2.2. Let $A$ be a nonempty subset of a $\Gamma$-semigroup $M$. Then

$$(A)_q = A \cup (M \Gamma A \cap A \Gamma M).$$

Proof. Let $A$ be a nonempty subset of a $\Gamma$-semigroup $M$.

Let $Q = A \cup (M \Gamma A \cap A \Gamma M)$. It is easy to see that $A \subseteq Q$. We have that $M \Gamma Q \cap Q \Gamma M = (A \cup (M \Gamma A \cap A \Gamma M)) \cap (A \cup (M \Gamma A \cap A \Gamma M)) \subseteq Q \Gamma M \cup A \Gamma M \subseteq Q \Gamma M \cup A \Gamma M \subseteq Q \Gamma M \cup A \Gamma M \subseteq Q$. Therefore, $Q$ is a quasi-$\Gamma$-ideal of $M$.

Let $C$ be any quasi-$\Gamma$-ideal of $M$ containing $A$. Since $C$ is a quasi-$\Gamma$-ideal of $M$ and $A \subseteq C$, $M \Gamma A \cap A \Gamma M \subseteq C$. Therefore, $Q = A \cup (M \Gamma A \cap A \Gamma M) \subseteq C$.

Hence, $Q$ is the smallest quasi-$\Gamma$-ideal of $M$ containing $A$. Therefore, $\{ A \}_q = A \cup (M \Gamma A \cap A \Gamma M)$, as required.

Example 2.2. Let $N$ be the set of natural integers and $\Gamma = \{ 5 \}$. Then $N$ is a $\Gamma$-semigroup under usual addition.

(i) Let $A = \{ 2 \}$. We have that $\{ A \} = \{ 2 \} \cup \{ 8, 9, 10, \ldots \}$.

(ii) Let $A = \{ 3, 4 \}$. We have that $\{ A \} = \{ 3, 4 \} \cup \{ 9, 10, 11, \ldots \}$.

Let $M$ be a $\Gamma$-semigroup. A sub-$\Gamma$-semigroup $L$ of $M$ is called a left $\Gamma$-ideal of $M$ if $MTL \subseteq L$ and a sub-$\Gamma$-semigroup $R$ of $M$ is called a right $\Gamma$-ideal of $M$ if $RTL \subseteq R$. The following theorem is true.

Theorem 2.3. Let $L$ be a $\Gamma$-semigroup. Let $L$ and $R$ be any left $\Gamma$-ideal and any right $\Gamma$-ideal of a $\Gamma$-semigroup $M$, respectively. Then $L \cap R$ is a quasi-$\Gamma$-ideal of $M$.

Proof. Let $L$ and $R$ be any left $\Gamma$-ideal and any right $\Gamma$-ideal of a $\Gamma$-semigroup $M$, respectively. By properties of $L$ and $R$, we have $RTL \subseteq L \cap R$. This implies that $L \cap R$ is a nonempty set. We have that

$$(L \cap R) \cap (L \cap R) \subseteq L \cap R \subseteq (L \cap R) \cap (L \cap R) \subseteq (L \cap R) \cap (L \cap R).$$

Hence, $L \cap R$ is a quasi-$\Gamma$-ideal of $M$.

Theorem 2.4. Every quasi-$\Gamma$-ideal $Q$ of a $\Gamma$-semigroup $M$ is the intersection of a left $\Gamma$-ideal and a right $\Gamma$-ideal of $M$.

Proof. Let $Q$ be any quasi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$. Let $L = Q \cup M \Gamma Q$ and $R = Q \cup Q \Gamma M$.

Then $MGL = MGL(Q \cup M \Gamma Q) = MGL \cup MGL \Gamma \subseteq M \Gamma Q \subseteq L$ and $RGM = RGM(Q \cup Q \Gamma M) \subseteq Q \Gamma M \cup Q \Gamma M \subseteq Q \Gamma M \subseteq R$. Then $L$ and $R$ is a left $\Gamma$-ideal and a right $\Gamma$-ideal of $M$, respectively.

Next, we claim that $Q = L \cap R$. It is easy to see that $Q \subseteq (Q \cup M \Gamma Q) \cap (Q \cup Q \Gamma M) \subseteq L \cap R$. Conversely, $L \cap R = (Q \cup M \Gamma Q) \cap (Q \cup Q \Gamma M) \subseteq Q \cup (M \Gamma Q \cup Q \Gamma M) \subseteq Q$. Hence, $Q = L \cap R$.

Let $M$ be a $\Gamma$-semigroup. $M$ is called a quasi-simple
By assumption, \( M \Gamma m \cap m \Gamma M = M \) for all \( m \in M \).

**Example 2.3.** Let \( G \) be a group and \( \Gamma = \{ e \} \). It is easy to see that \( \Gamma \) is a unique quasi-\( \Gamma \)-ideal of \( G \) under the usual binary operation. Then \( G \) is a quasi-simple \( \Gamma \)-semigroup.

**Theorem 2.5.** Let \( M \) be a \( \Gamma \)-semigroup. Then \( M \) is a quasi-simple \( \Gamma \)-semigroup if and only if \( M \Gamma m \cap m \Gamma M = M \) for all \( m \in M \).

**Proof.** Let \( M \) be a \( \Gamma \)-semigroup.

The proof of \((\rightarrow)\): Assume that \( M \) is a quasi-simple \( \Gamma \)-semigroup. Take any \( m \in M \). First, we claim that \( M \Gamma m \cap m \Gamma M \) is a quasi-ideal of \( M \). We have that \( m \Gamma m \in M \Gamma m \cap m \Gamma M \), this implies \( M \Gamma m \cap m \Gamma M \) is a nonempty set. Moreover, \( M \Gamma (M \Gamma m \cap m \Gamma M) \cap (M \Gamma m \cap m \Gamma M) \Gamma M \subseteq M \Gamma (M \Gamma m \cap m \Gamma M) \cap (m \Gamma M) \subseteq M \Gamma m \cap m \Gamma M \). Therefore, \( M \Gamma m \cap m \Gamma M \) is a quasi-\( \Gamma \)-ideal of \( M \). Since \( M \) is a quasi-simple \( \Gamma \)-semigroup, \( M \Gamma m \cap m \Gamma M = M \).

The proof of \((\leftarrow)\): Assume that \( M \Gamma m \cap m \Gamma M = M \) for all \( m \in M \). Let \( Q \) be a quasi-\( \Gamma \)-ideal of \( M \) and \( q \in Q \). By assumption, \( M = M \Gamma q \cap q \Gamma M \). Since \( Q \) is a quasi-\( \Gamma \)-ideal of \( M \), \( M = M \Gamma q \cap q \Gamma M \subseteq M \Gamma Q \cap Q \Gamma M \subseteq Q \). Therefore \( Q = M \). Hence, \( M \) is a quasi-simple \( \Gamma \)-semigroup.

**Theorem 2.6.** Let \( M \) be a \( \Gamma \)-semigroup and \( Q \) a quasi-\( \Gamma \)-ideal of \( M \). If \( Q \) is a quasi-simple \( \Gamma \)-semigroup, then \( Q \) is a minimal quasi-\( \Gamma \)-ideal of \( M \).

**Proof.** Suppose \( M \) be a \( \Gamma \)-semigroup and \( Q \) a quasi-\( \Gamma \)-ideal of \( M \). Assume that \( Q \) is a quasi-simple \( \Gamma \)-semigroup. Let \( C \) be a quasi-\( \Gamma \)-ideal of \( M \) such that \( C \subseteq Q \). Then \( Q \Gamma C \cap C \Gamma Q \subseteq M \Gamma C \cap C \Gamma M \subseteq C \). Therefore, \( C \) is a quasi-\( \Gamma \)-ideal of \( Q \). Since \( Q \) is a quasi-simple \( \Gamma \)-semigroup, \( C = Q \). Then \( Q \) is a minimal quasi-\( \Gamma \)-ideal of \( M \).

**Acknowledgement**

The author would like to thank the referees for their useful and helpful suggestions.

**References**