Some Geometric properties in Orlicz- Cesaro Spaces

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ABSTRACT: On the Orlicz- Cesaro sequence spaces (ces<φ>) which are defined by using Orlicz function φ, we show that the space ces<φ> equipped with both Amemiya and Luxemburg norms possesses uniform Opial property and uniform Kadec-Klee property if φ satisfy the Ω-condition.

KEYWORDS: Orlicz-Cesaro sequence spaces, uniform Kadec-Klee property, uniform Opial property, Amemiya norm, Luxemburg norm.

INTRODUCTION

In the whole paper N and R stand for the sets of natural numbers and of real numbers, respectively. The space of all real sequences is denoted by ℓ₀. Let (X,∥·∥) be a real normed space and B(X)/(S(X)) be the closed unit ball (the unit sphere) of X.

A Banach space (X,∥·∥) which is a subspace of ℓ₀ is said to be a Kothe sequence space, if:

(i) for any x ∈ ℓ₀ and y ∈ X such that |x(i)| ≤ |y(i)| for all i ∈ N, we have x ∈ X and ∥x∥ ≤ ∥y∥,
(ii) there is x ∈ X with x(i) ≠ 0 for all i ∈ N.

An element x from a Kothe sequence space X is called order continuous if for any sequence (xₙ) in X⁺ (the positive cone of X) such that xₙ ≤ |x| for all n ∈ N and xₙ → x coordinatewise, we have ∥xₙ∥ → 0.

A Kothe sequence space X is said to be order continuous if any x ∈ X is order continuous. It is easy to see that x ∈ X is order continuous if and only if ∥(0,0,...,0,x(n+1),x(n+2),...)| → 0 as n → ∞.

A Banach space X is said to have the Kadec-Klee property (see (4)) if for each ε > 0 there exists τ > 0 such that for every weak null sequence (xₙ) in S(X) and x ∈ X with ∥x∥ ≥ ε the following inequality holds:

1 + τε ≤ lim inf ∥xₙ + x∥.

A Banach space X is said to have the uniform Kadec-Klee property (write (UKK) for short) if for every ε > 0 there exists τ > 0 such that for every weak null sequence (xₙ) in S(X) and x ∈ X with ∥x∥ ≥ ε the following inequality holds:

1 + τε ≤ lim inf ∥xₙ + x∥.

A Banach space X is said to have the Opial property (see (3)) if for any weakly null sequence (xₙ) and every x ≠ 0 in X, we have

lim inf ∥xₙ∥ < lim inf ∥xₙ + x∥.

A Banach space X is said to have the uniform Opial property (see (4)) if for each ε > 0 there exists τ > 0 such that for any weak null sequence (xₙ) in S(X) and x ∈ X with ∥x∥ ≥ ε the following inequality holds:

1 + τε ≤ lim inf ∥xₙ + x∥.

For a real vector space X, a function M : X → [0,∞] is called a modular if it satisfies the following conditions:

(i) M(x) = 0 if and only if x = 0,
(ii) M(αx) = αM(x) for all scalar α with |α| = 1,
(iii) M(αx + βy) ≤ αM(x) + βM(y), for all x, y ∈ X and all α, β ≥ 0 with α + β = 1.

The modular M is called convex if

(iv) M(αx + βy) ≤ αM(x) + βM(y), for all x, y ∈ X and all α, β ≥ 0 with α + β = 1.

For any modular M on X, the space Xₘ = {x ∈ X : M(λx) → 0 as λ → 0}, is called the modular space.

A sequence (xₙ) of elements of Xₘ is called modular convergent to x ∈ Xₘ if there exists a λ > 0 such that M(λ(xₙ) - x) → 0, as n → ∞.

If M is a convex modular, the function

∥x∥ = inf {A > 0 : M(Ax) ≤ 1},

and

∥x∥ₘ = inf {1 + M(Ax) : A > 0},

are two norms on Xₘ, which are called the Luxemburg norm and the Amemiya norm, respectively. In addition, ∥x∥ₘ ≤ 2∥x∥ for all x ∈ Xₘ (see [5]).

Theorem 1.1 Let (xₙ) ⊂ Xₘ then ∥xₙ∥ → 0 (or equivalently ∥xₙ∥ₘ → 0) if and only if M(λ(xₙ)) → 0, as n → ∞, for every λ > 0.
Proof. See [6, Theorem 1.3(a)].

A modular $\Omega$ is said to satisfy the $\Delta_2$-condition ($\Omega \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that
$$\Omega(2x) \leq K\Omega(x) + \varepsilon$$
for all $x \in X_\Omega$ with $\Omega(x) \leq a$.

If $\Omega$ satisfies the $\Delta_2$-condition for all $a > 0$ with $K \geq 2$ dependent on $a$, we say that $\Omega$ satisfies the strong $\Delta_2$-condition ($\Omega \in \Delta_2^s$).

**Theorem 1.2** Convergences in norm and in modular are equivalent in $X_\Omega$ if $\Omega \in \Delta_2$.

**Proof.** See [7, Lemma 2.3].

**Theorem 1.3** If $\Omega \in \Delta_2^s$ then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\Omega(u + v) \leq L \Omega(u) \leq \delta$$
equivalently, whenever $u, v \in X_\Omega$ with $\Omega(u) \leq L$ and $\Omega(v) \leq \delta$.

**Proof.** See [7, Lemma 2.4].

**Theorem 1.4** If $\Omega \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $\Omega(\|x\|) \geq 1 + \varepsilon$.

**Proof.** See [7, Lemma 2.2].

An Orlicz function $\Phi$ is defined as
$$\Phi(x) = \int_0^x \phi(t)\,dt$$
in $L^1(0, \infty)$, where $\phi$ is a non-negative, non-decreasing function.

The Orlicz sequence space, $\ell_\Phi$, where $\Phi$ is an Orlicz function is defined as
$$\ell_\Phi = \{x \in \ell^1 : \phi\Omega(\|x\|) < \infty\}$$
where $\phi\Omega(\|x\|) = \sum_{n=1}^\infty \phi(\|x_n\|)$.

An Orlicz function $\Phi$ is said to satisfy the $\Delta_2$-condition (we will write $\Phi \in \Delta_2^s$) if there exist constants $K \geq 2$ and $a > 0$ such that the inequality $\Phi(2x) \leq K\Phi(x)$ holds for every $x \in \ell_\Phi$ satisfying $\|x\| \leq a$.

For $1 < p < \infty$, the Cesaro sequence space (write $\ell_\Phi^{\text{ces}}$, for short) is defined by
$$\ell_\Phi^{\text{ces}} = \{x \in \ell^1 : \sum_{n=1}^\infty \frac{1}{n}\phi(\|x_n\|) < \infty\},$$
equipped with the norm
$$\|x\| = \left(\sum_{n=1}^\infty \frac{1}{n}\phi(\|x_n\|)^p\right)^{1/p}$$
(1.1)

This space was first introduced by Shiue [9]. It is useful in the theory of Matrix operators and others (see [10] and [11]). Some geometric properties of the Cesaro sequence spaces $\ell_\Phi^{\text{ces}}$ were studied by many authors.

For an Orlicz function $\Phi$ the Orlicz- Cesaro sequence space $\ell_\Phi^{\text{ces}}$, is defined by
$$\ell_\Phi^{\text{ces}} = \{x \in \ell^1 : \rho_\Phi(\lambda x) < \infty, \exists \lambda > 0\},$$
where
$$\rho_\Phi(x) = \sum_{n=1}^\infty \Phi\left(\frac{1}{n}\sum_{i=1}^n |x(i)|\right),$$
and $\Phi$ is a convex modular on $\ell_\Phi^{\text{ces}}$. The subspace $E_\Phi$ of $\ell_\Phi^{\text{ces}}$ is defined by
$$E_\Phi = \{x \in \ell^1 : \rho_\Phi(\lambda x) < \infty, \forall \lambda > 0\}.$$

It is worth noting that if $\Phi \in \Delta_2^s$, then $\rho_\Phi \in \Delta_2^s$ and $E_\Phi = E_\Phi^{\text{ces}}$.

To simplify notations, we put $E_\Phi = (E_\Phi, \rho_\Phi)$ and $E_\Phi^{\text{ces}} = (E_\Phi^{\text{ces}}, \rho_\Phi^{\text{ces}})$. In the case when $\Phi(t) = t^p$, $p > 1$ the Orlicz- Cesaro sequence space $\ell_\Phi^{\text{ces}}$ becomes the Cesaro sequence space $\ell_\Phi^{\text{ces}}$, and the Luxemburg norm is that one defined by (1.1).

From now on, for $x = (x(1), x(2), \ldots, x(1), x(1), x(1))$, and
$$x_n = (0, 0, \ldots, x(i + 1), x(i + 2), x(i + 3), \ldots),$$
and
$$\supp x = \{i \in \mathbb{N} : x(i) \neq 0\}.$$

**Results**

We first give an important fact for $\|x\|_\Phi$ on $\ell_\Phi^{\text{ces}}$.

**Lemma 2.1** If $\Phi$ is an $\mathcal{N}$-function, then for each $x \in \ell_\Phi^{\text{ces}}$ there exists $k \in \mathbb{R}$ such that
$$\|x\|_\Phi = \frac{1}{k}(\lambda + \rho_\Phi(\lambda x)).$$

**Proof.** For each $x = (x(1), \ldots, x(n)) \in \ell_\Phi^{\text{ces}}$, we have
$$x = \left(\frac{1}{n}\sum_{i=1}^n |x(i)|\right)^n \in \ell_\Phi^{\text{ces}}.$$
and \( k_n \to \infty \), then \( x = 0 \).

**Proof.** For each \( n \in \mathbb{N}, \eta > 0 \), put 
\[
G_{(n, \eta)} = \left\{ i \in \mathbb{N} : \frac{1}{i} \sum_{i=1}^{n} |x_n(i)| \geq \eta \right\}.
\]
First, we claim that for each \( \eta > 0, G_{(n, \eta)} = \emptyset \) for all large \( n \in \mathbb{N} \). Otherwise, without loss of generality, we may assume that \( G_{(n, \eta)} \neq \emptyset \) for all \( n \in \mathbb{N} \) and for some \( \eta > 0 \). Then
\[
\|x_n\|_A = \frac{1}{k_n} \left( 1 + \rho_\phi(k_n x_n) \right) \geq \frac{\Phi(k_n x_n)}{k_n} \quad \left( i \in G_{(n, \eta)} \right).
\]

By applying the assumption that \( \Phi \) is an \( \mathcal{N}^r \)-function, we obtain \( \|x_n\|_A \to \infty \), which contradicts to the fact that \( \{x_n\} \) is bounded, hence we have the claim. By the claim, we have \( \frac{1}{i} \sum_{i=1}^{n} |x_n(i)| \to 0 \) as \( n \to \infty \) for all \( i \in \mathbb{N} \). This implies that \( x_n(i) \to 0 \) as \( n \to \infty \) for all \( i \in \mathbb{N} \). Since \( x_n \rightharpoonup x \), we have \( x_n(i) \to x(i) \) for all \( i \in \mathbb{N} \), so it follows that \( x(i) = 0 \) for all \( i \in \mathbb{N} \).

**Lemma 2.3** For any Orlicz function \( \Phi \), we have 
\[
E < \{ x \in \mathcal{C} \mathcal{S}_\Phi : \|x - x_0\|_A \leq 0 \}.
\]

**Proof.** Write 
\[
A = \{ x \in \mathcal{C} \mathcal{S}_\Phi : \|x - x_0\|_A \leq \varepsilon \}. \quad \text{Let } x \in E_\Phi \text{ and } \varepsilon > 0 . \quad \text{Since } x \in E_\Phi, \text{ there exists } i_0 \in \mathbb{N} \text{ such that } \rho_\phi \left( \|x - x_i\|_A \right) < \varepsilon \quad \text{for all } i > i_0 . \quad \text{Therefore, by the definition of } \| \cdot \|_A \text{ we have }
\]
\[
\varepsilon^{-1} \|x - x_i\|_A \leq 1 + \rho_\phi \left( \|x - x_i\|_A \right) < 1 + \varepsilon \quad \text{for all } i > i_0 .
\]
This yields \( \|x - x_i\|_A \to 0 \) as \( i \to \infty \) since \( \varepsilon \) is arbitrary. Hence \( x \in A \), proving the Lemma.

**Theorem 2.4** The space \( \mathcal{C} \mathcal{S}_\Phi \) is (UKK) if \( \Phi \) is an \( \mathcal{N}^r \)-function which satisfies the \( \delta^2 \)-condition.

**Proof.** For a given \( \varepsilon > 0 \), by Theorem 1.2 there exists \( \delta \in (0,1) \) such that \( \|x\|_A \geq \varepsilon \) implies \( \Phi \left( \|x\|_A \right) \geq \delta \). Given \( x_n \in \mathcal{E} \mathcal{C} \mathcal{S}_\Phi \), \( x_n \rightharpoonup x \) weakly and \( \|x_n - x_m\|_A \geq \varepsilon (n \neq m) \), we shall complete the proof by showing that \( \|x\|_A \leq 1 - \delta \). Indeed, if \( x = 0 \), then it is clear. So, we assume \( x \neq 0 \). In this case, by Proposition 2.2 we have that \( \{k_n\} \) is bounded, where \( k_n \in K \{x_n\} \). Passing to a subsequence if necessary we may assume that \( k_n \to k \) for some \( k > 0 \). Since \( \Phi \in \delta^2 \), Lemma 2.3 assures that there exists \( j \in \mathbb{N} \) such that
\[
\|x_n - x_m\|_A \leq \varepsilon \quad \text{for all } n, m. \quad \text{This gives } \|x_n - x_m\|_A \leq \frac{\varepsilon}{2} \quad \text{for all } n, m \geq n_0 . \quad \text{If } k_n \leq 1, \text{ then it is clear. So, we assume } k_n > 1 . \quad \text{By using the convexity of } \Phi \text{ and the inequality } \Phi(a + b) \geq \Phi(a) + \Phi(b), \text{ we have }
\]
\[
1 - 2\delta \geq \|x_n - x_m\|_A - \rho_\phi \left( \|x_n - x_m\|_A \right) \geq \frac{1}{k_n} \rho_\phi \left( k_n x_n \right) \geq \|x_n\|_A \geq 1 - \delta ,
\]

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hence \( \|x\|_\alpha \leq 1 - \delta \).

**Theorem 2.5** If \( \Phi \) is an \( N' \)-function which satisfies \( \delta \)-condition, then \( c_0 \) has the uniform Opial property.

**Proof.** Take any \( \varepsilon > 0 \) and \( x \in c_0 \) with \( \|x\| \geq \varepsilon \). Let \( (x_n) \) be weakly null sequence in \( S(c_0) \). By \( \Phi \in \delta_2 \), and Theorem 1.2, there is \( \xi \in (0,1) \) independent of \( x \) such that \( \rho \left( \frac{x}{\|x\|} \right) > \xi \). Also, by \( \Phi \in \delta_2 \), we have \( c_0 = E_\rho \). By Lemma 2.3, \( x \) is an order continuous element, this allows us to find \( j_0 \in \mathbb{N} \) such that

\[
\xi \leq \sum_{j=1}^{\infty} \Phi \left( \frac{1}{j} \sum_{i=j}^{\infty} |x(i)| \right) < \frac{\xi}{8}.
\]

It follows that

\[
\xi \leq \sum_{j=1}^{\infty} \Phi \left( \frac{1}{j} \sum_{i=j}^{\infty} |x(i)| \right) + \sum_{j=1}^{\infty} \frac{\xi}{8} < \frac{7\xi}{8}.
\]

From \( x_n \to \infty \), we have \( x_n(i) \to 0 \) for all \( i \in \mathbb{N} \), which implies that \( \rho \left( x_n \right) \to 0 \). By Theorem 1.2 we have \( \|x_n\| \to 0 \), so there exists \( n_0 \in \mathbb{N} \) such that \( \|x_n\| < \frac{\xi}{4} \) for all \( n > n_0 \).

Therefore,

\[
\|x + x_n\| = \|x + x_n - x + x_n\|_{\alpha} \geq 1 - \frac{\xi}{4}.
\]

Hence \( x_n \to x \) weakly. Since \( \Phi \) is an \( N' \)-function, by Lemma 2.1 there exists \( k_0 > 0 \) such that

\[
\|x_n + x_{n+k-1}\| = \frac{1}{k_0} \left( 1 + \rho \left( k_0 \left( x_n + x_{n+k-1} \right) \right) \right) \leq \frac{1}{k_0} \left( 1 + \rho \left( k_0 x_n \right) \right).
\]

This together with (2.2) and the fact that \( \rho \left( y + z \right) \geq \rho \left( y \right) + \rho \left( z \right) \) if \( \text{supp} \, y \cap \text{supp} \, z = \emptyset \), we have

\[
\|x + x_n\| \geq \frac{1}{k_0} \left( 1 + \rho \left( k_0 x_n \right) \right) - \frac{\xi}{2} \geq \|x_n + x_{n+k-1}\| - \frac{\xi}{2}.
\]

We may assume without loss of generality that \( k_n \geq \frac{1}{2} \). Since \( 2k_n \geq 1 \), by convexity of Orlicz function \( \Phi \) we have that \( \rho \left( k_n x_n \right) \leq 2 \rho \left( k_n x_n \right) \leq 2 \rho \left( x_n \right) \leq 2 \rho \left( \frac{x_n}{2} \right) = \frac{\xi}{2} \). Thus inequalities (2.1) and (2.3) imply that

\[
\|x + x_n\| \geq \|x_n + x_{n+k-1}\| + 2 \rho \left( \frac{x_n}{2} \right) = \frac{\xi}{2}.
\]

Theorem 2.6 If \( \Phi \) is an Orlicz function which satisfies \( \delta_2 \)-condition, then \( c_0 \) has the uniform Opial property.

**Proof.** Take any \( \varepsilon > 0 \) and \( x \in c_0 \) with \( \|x\| \geq \varepsilon \). Let \( (x_n) \) be weakly null sequence in \( S(c_0) \). By \( \Phi \in \delta_2 \), we have \( \rho \in \delta_2 \). Thus by Theorem 1.2, there is \( \eta \in (0,1) \) independent of \( x \) such that \( \eta < \rho \left( x \right) \). Also, by \( \Phi \in \delta_2 \), Theorem 1.3 asserts that there exists \( \eta \in (0,\eta) \) such that

\[
\rho \left( y + z \right) \geq \rho \left( y \right) + \rho \left( z \right) \text{ if } \text{supp} \, y \cap \text{supp} \, z = \emptyset.
\]

This together with the assumption that \( x_n \to x \), there exists \( n_0 \in \mathbb{N} \) such that \( \frac{3n}{4} \leq \sum_{j=1}^{\infty} \Phi \left( \frac{1}{j} \sum_{i=1}^{\infty} |x(i)| \right) \). (2.6)

for all \( n > n_0 \), since the weak convergence implies the coordinatewise convergence. Again by \( x_n \to x \), there exists \( n_0 \in \mathbb{N} \) such that \( \rho \left( x_n \right) < \eta \) for all \( n > n_0 \). Hence, \( \frac{1}{n} - \frac{n}{4} = \rho \left( x_n \right) - \frac{n}{4} \leq \rho \left( x_{n+k-1} \right) \),

which deduces \( \liminf_{n \to \infty} \|x + x_n\|_{\alpha} \geq 1 + \frac{\xi}{8} \).
for all $n > n_1$. This together with (2.4), (2.5) and (2.6) imply that for any $n > n_1$, 
\[
\rho_\phi(x_n + x) \geq \sum_{j=n_1}^{\infty} \Phi \left( \frac{1}{j} \sum_{i=1}^{j} x_i(i) + x(i) \right) + \sum_{j=n_1}^{\infty} \Phi \left( \frac{1}{j} \sum_{i=1}^{j} x_i(i) + x(i) \right) 
\geq \frac{3\eta}{4} + \frac{\sum_{j=n_1}^{\infty} \Phi \left( \frac{1}{j} \sum_{i=1}^{j} x_i(i) + x(i) \right)}{\eta} - \frac{\eta}{4} = 1 + \frac{\eta}{4}.
\]
By $\rho_\phi \in \Delta'_2$, and by Theorem 1.4, there is $\tau$ depending on $\eta$ only such that $\|x_n + x\| \geq 1 + \tau$.

**Corollary 2.7** ([12, Theorem 2]) For any $1 < p < \infty$, the space $c_0^p$ has the uniform Opial property.

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**REFERENCES**