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On Generalized Rearick Logarithm

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Abstract: In this paper, we introduce the generalized Rearick logarithm and give its series representation. Moreover, we associate this logarithmic operator with other additive functions.

Introduction

By an *arithmetic function*, we mean a complex-valued function whose domain is the set of positive integers, **N**. We define the addition and convolution of two arithmetic functions f and g, respectively, by

$$(f+g)(n) = f(n)+g(n),\ (f*g)(n) = \sum_{ij=n} f(i)g(j).$$

It is well known (see e.g. [1], [2], [3], [4], [7]) that the set $(A_C, +, *)$ of all arithmetic functions is a unique factorization domain with the arithmetic function

$$I(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

being its convolution identity.

For notational conveniences, let

$A'_C = \{f \in A_C : f(1) \in \mathbf{R}\}$

$P_C := \{f \in A_C : f(1) > 0\}$, the so-called positive elements of A_C

$U_1 = \{f \in A_C : f(1) = 1\}$, the so-called normalized units of A_C.

To facilitate later discussion, we recall some facts about derivation over A_C, which is a function $d : A_C \to A_C$ satisfying

$$d(f*g) = df*g + f*dg,\ d(rf+sg) = rdf+sdg,$$

for all $f, g \in A_C$, and for all $r, s \in \mathbf{C}$.

Recall that $a \in A_C$ is *completely additive* if $a(mn) = a(m)+a(n)$ ($\forall\, m, n \in \mathbf{N}$).

The derivation d_a associated with this additive $a \in A_C$ is defined by

$$d_a f(n) = f(n)a(n)\ (n \in \mathbf{N}).$$

Generalized Rearick Logarithm

Definition 2.1 *Let $a \in A_C$ be a completely additive arithmetic function for which $a(n) \neq 0$ ($\forall\, n>1$), and d_a its associated derivation. The logarithmic operator (associated with a) is the map $\mathrm{Log} : P_C \to A'_C$, defined by $(\mathrm{Log} f)(1) = \log f(1)$, where the right-hand side denotes the real logarithmic value, and*

$$\begin{aligned}\mathrm{Log} f(n) &= \frac{1}{a(n)}\,(d_a f * f^{-1})(n) \\ &= \frac{1}{a(n)} \sum_{d|n} f(d) f^{-1}\!\left(\frac{n}{d}\right) a(d) \quad (n>1).\end{aligned}$$

Remark. The logarithmic operator of Rearick ([5], [6]) uses the additive function $\log n$ in place of $a(n)$.

Theorem 2.2 *For all $f, g \in P_C$, we have*

$$\mathrm{Log}(f*g) = \mathrm{Log} f + \mathrm{Log} g.$$

Proof. If $n = 1$, we see that

$$\begin{aligned}\mathrm{Log}(f*g)(1) &= \log((f(1)g(1)) = \log f(1) + \log g(1) \\ &= \mathrm{Log} f(1) + \mathrm{Log} g(1).\end{aligned}$$

For $n > 1$, we have

$$\begin{aligned}\mathrm{Log}(f*g)(n) &= \frac{1}{a(n)}(d_a(f*g)*(f*g)^{-1})(n) \\ &= \frac{1}{a(n)}[(f*d_a g + g*d_a f)*g^{-1}*f^{-1}](n) \\ &= \frac{1}{a(n)}(g^{-1}*d_a g + f^{-1}*d_a f)(n) \\ &= \mathrm{Log} f(n) + \mathrm{Log} g(n).\end{aligned}$$

Theorem 2.3 *For each $h \in A'_C$, there is a unique $f \in P_C$ such that h = Log$f$.*

Proof. We proceed inductively. Define $f(1) = \exp h(1)$. Assume that $f(k)$ has been defined for all $k < n$, where $n > 1$. The value $f^{-1}(k)$ are recursively determined by the relation $\sum_{d|k} f^{-1}(d) f(\frac{k}{d}) = 1$ if k = 1, and 0 elsewhere. This gives us a triangular system which can be solved for the unknowns $f^{-1}(k)$. With $h(n)$ prescribed and $f(k)$, $f^{-1}(k)$ determined for all $k < n$, we can define $f(n)$ by solving for the term corresponding to $d = n$ in the equation

$$h(n) = \frac{1}{a(n)} \sum_{d|n} f(d) f^{-1}(\frac{n}{d}) a(d).$$

The term containing $f^{-1}(n)$ is suppressed by the presence of the factor $a(1) = 0$, and all other terms are known. This inductive process allows us to construct a function f such that it satisfies the condition h = Log$f$, and at the same time it guarantees the uniqueness of this f, since the value of $f(n)$ is uniquely determined at the n[th] step.

Theorem 2.4 *Let $f \in P_C$. Then f is multiplicative if and only if Log$f(n) = 0$ whenever n is not a prime (positive) power.*

Proof. Assume that f is multiplicative. Then $f(1) = 1$ and so Log$f(1)$ = log$f(1) = 0$. Let N be a positive integer which is not a prime power. Then there are positive integers m, n, both of them greater than 1 and $(m,n) = 1$ such that $N = mn$. We will show that Log$f(N) = 0$. We have

$$\text{Log}f(N) = \frac{1}{a(N)} \sum_{d|mn} f(d) f^{-1}(\frac{mn}{d}) a(d)$$

$$= \frac{1}{a(N)} \sum_{d_1|m} \sum_{d_2|n} f(d_1) f(d_2) f^{-1}(\frac{m}{d_1}) f^{-1}(\frac{n}{d_2})(a(d_1) + a(d_2)),$$

where we have decomposed each d uniquely into the product of a divisor d_1 of m and a divisor d_2 of n. Thus

$$\text{Log}f(N) = \frac{I(n)}{a(N)} \sum_{d_1|m} f(d_1) f^{-1}(\frac{m}{d_1}) a(d_1) + \frac{I(m)}{a(N)} \sum_{d_2|n} f(d_2) f^{-1}(\frac{n}{d_2}) a(d_2)$$

$$= 0 \text{ (since } I(n) = I(m) = 0).$$

Conversely, suppose that Log$f(n) = 0$ whenever n is not a prime (positive) power. Then Log$f(1) = 0$, so $f(1) = 1$. For $n > 1$, we let $g \in P_C$ be defined by $g(1) = 1$, $g(n) = \prod_{p|n} f(p^\nu)$ where $p^\nu | n, p^{\nu+1} \nmid n$.

Clearly, g is multiplicative. It remains to show that $f = g$. Observe that $f(n) = g(n)$ and $f^{-1}(n) = g^{-1}(n)$ whenever n is a prime power. From definition, we have Log$f(n)$ = Log$g(n)$ if n is a prime power. Since g is multiplicative, the previous half of this theorem shows that Log$g(m) = 0$ if m is not a prime power. Hence, Log$f(n)$ = Log$g(n)$ for all $n \in \mathbf{N}$ and therefore $f = g$ by Theorem 2.3, proving our Theorem.

SERIES REPRESENTATION

Let $F = \{u_j : j = 1, 2, \ldots\}$ be a sequence of non-negative integers u_j such that $u_1 = 0$ and only finitely many u_j are non-zero. We call the sequence F a *factorization*. Two factorizations are considered *equal* if and only if the sequences are identical. If $\prod_{j=1}^{\infty} j^{u_j} = n$, we call F a *factorization* of n, and also say that n is the *index* of F and we write $i(F) = \prod_{j=1}^{\infty} j^{u_j} = n$. If $n > 1$, one particular factorization of n is the *identity factorization*, in which $u_n = 1$ and $u_j = 0$ if $j \neq n$. When $n = 1$, the only factorization is the *zero factorization* O, in which all $u_j = 0$. We define the *height* h_F of a factorization F by $h_F = \sum_j u_j$, and also define the partial ordering of factorizations as follows: $F' \leq F$ if $u_j' \leq u_j$ for all j. By $F' < F$ we mean $F' \leq F$ and $F' \neq F$. The sum of any two factorizations is defined as the termwise sum of the sequences, i.e., $F + F' = \{u_j + u_j'\}$. If $F' \leq F$, we define the difference $F - F' = \{u_j - u_j'\}$. Clearly, $i(F+F') = i(F)i(F')$ and $i(F-F') = \frac{i(F)}{i(F')}$. If F is a factorization and $f \in A_c$, we define

$$f^F = \prod_j (f(j))^{u_j}$$

If we adopt the convention that $0^0 = 1$, then the product may be considered to be extended for all $j \geq 1$, with only finitely many factors different from 1.

If $F = \{u_j\}$ is a factorization, set $F! = \prod_j (u_j!)$. If $F' \leq F$, we define $\binom{F}{F'} = \frac{F!}{F'!(F-F')!}$.

If r is any real number, set $\binom{r}{O} = 1$, and if $F > O$, define $\binom{r}{F} = \frac{r(r-1)\ldots(r-h+1)}{F!}$ where $h = h_F$ is the height of F.

Theorem 3.1 *For all $f \in U_1$,*

$$\mathrm{Log}f(n) = \sum_{i(F)=n,h>0} \binom{h}{F} \frac{(-1)^{h+1} f^F}{h}$$

where $h = h_F$ is the height of F.

Proof. If $n = 1$, the index set of the above sum is empty and the sum is understood to have the value zero, which is compatible with $\mathrm{Log}f(1) = \log 1 = 0$. For each $n > 1$, by definition we have

$$\mathrm{Log}f(n)a(n) = d_a \mathrm{Log}f(n) = (d_a f * f^{-1})(n) = (d_a f * \sum \binom{-1}{F} f^F)(n)$$

(by Theorem 3 of [6])

$$= \sum_{i(F)=n} c(F) f^F \quad \text{(by Corollary 2a of [6])}$$

where $c(F) = \sum_{J \le F} \binom{-1}{F-J} a(i(J))$. Since $F > O \Leftrightarrow h = h_F > 0$, Lemma 3 of [6] yields $c(F) = (-1)^{h+1} \binom{h}{F} \frac{a(n)}{h}$, and the result follows.

Theorem 3.2 *Let $f \in P_C$ and $\bar{f}(n) = \frac{f(n)}{f(1)}$ $(n \in \mathbf{N})$.*
Then $\mathrm{Log}f(1) = \log f(1)$ *and*

$$\mathrm{Log}f(n) = \sum_{i(F)=n,h>0} \binom{h}{F} \frac{(-1)^{h+1} \bar{f}^F}{h} \quad (n > 1),$$

where $h = h_F$ is the height of F.

Proof. The value at $n = 1$ follows from the definition. Since $\bar{f}(1) = \frac{f(1)}{f(1)} = 1$, we have $\bar{f} \in U_1$, and hence Theorem 3.1 gives

$$\mathrm{Log}\bar{f}(n) = \sum_{i(F)=n,h>0} \binom{h}{F} \frac{(-1)^{h+1} \bar{f}^F}{h} \quad (n > 1).$$

We now have

$$\begin{aligned}
a(n)\mathrm{Log}\bar{f}(n) &= d_a \mathrm{Log}\bar{f}(n) \qquad (n > 1) \\
&= \sum_{d|n} \bar{f}(d) \bar{f}^{-1}(\tfrac{n}{d}) a(d) \\
&= \sum_{d|n} \frac{f(d)}{f(1)} f^{-1}(\tfrac{n}{d}) f(1) a(d) \\
&\qquad (\text{using } \bar{f}^{-1}(\tfrac{n}{d}) = f^{-1}(\tfrac{n}{d}) f(1)) \\
&= \sum_{d|n} f(d) f^{-1}(\tfrac{n}{d}) a(d) \\
&= a(n)\mathrm{Log}f(n)
\end{aligned}$$

and the desired result follows from Theorem 3.1.

Logarithmic Operators Associated with Other Additive Functions

The logarithmic operator, defined in section 2, is associated with an additive function $a(n)$ which is non-zero for all $n > 1$. This is modified upon Rearick's operator where $a(n) = \log n$.

For a general additive function $\nu(n)$, a similar logarithmic operator can also be defined by dropping the normalized term $\nu(n)$ as follows:

$$\begin{aligned}
&L_\nu : P_C \to A'_C \\
&L_\nu f(1) = \log f(1) \\
&L_\nu f(n) = (d_\nu f * f^{-1})(n) = \sum_{d|n} f(d)\nu(d) f(\tfrac{n}{d}).
\end{aligned}$$

Theorem 4.1 (*i*) *For all $f, g \in P_C$, we have $L_\nu(f * g) = L_\nu f + L_\nu g$*

(*ii*) *Let $f \in P_C$. Then f is multiplicative if and only if $\mathrm{Log}f(n) = 0$ whenever n is not a prime (positive) power.*

Proof. The proof is similar to that of Theorems 2.2 and 2.4.

Theorem 4.2 *Let $f \in P_C$ and $\bar{f}(n) = \frac{f(n)}{f(1)}$ $(n \in \mathbf{N})$.*
Then for $n > 1$,

$$L_\nu f(n) = \nu(n) \sum_{i(F)=n} \binom{h}{F} \frac{(-1)^{h+1} \bar{f}^F}{h} \qquad (h = h_F).$$

Proof. We use the same argument as that given in Theorem 3.2.

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