ScienceAsia 26 (2000) : 39-48

Some Transcendental Elements in Positive Characteristic

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Received 22 Sep 1999
Accepted 17 Nov 1999

ABSTRACT Five transcendental elements in function fields of positive characteristic are constructed embracing those previously derived by Wade during 1941-43. The construction results from a careful analysis of the original works of Wade and indicates that this method and its associated technique is still worthy of consideration.

KEYWORDS: transcendental, characteristic.

INTRODUCTION

The study of transcendence in fields of nonzero characteristic is known to begin in 1941 through the paper of Wade.[12] Since then, various approaches, tools and results,see *eg* Amou,[1] Brownawell,[2] Dammme and Hellegouarch,[4] Denis,[5] Geijsel,[7,8] Goss,[10] Mathan,[11] Wade,[12-16] Yu,[17-23] have been investigated. Though the modern treatment of the subject via the concept of Drinfeld modules (Drinfeld[6]) is dominating the present day research, the old and classical ideas and approach of Wade in 1941 still prove to be a direct and a very powerful technique of establishing certain specific transecn-dence results as evidenced in some recent works of Damamme and Hellegouarch.[4] A close analysis of the works of Wade[12-16] reveals that the ideas consist of first assuming algebraicity, second finding multipliers, third separating appropriate expressions into the so-called integral and remainder parts, fourth estimating upper and lower bounds for the integral and remainder parts and fifth and finally, deriving a contradiction from the upper and lower bounds or the like. The main objective of this work is to substantiate this belief by proving five theorems generalizing corresponding earlier results of Wade[12-16] basing on a careful analysis of the original method of Wade mentioned above.

The following terminology and notation, see also,[9] are standard throughout the entire paper.

$F_q[x]$ the ring of polynomials over the Galois (finite) field F_q of characteristic p with p being prime, q being a power of p.

$F_q(x)$ the quotient field of $F_q[x]$.

deg or $|\,.\,|$ (or $|\,.\,|_\infty$) the nonarchimedean valuation (at ∞) normalized so that $|x| = q^{\deg x} = q$.

$F_q(x)_\infty$ the completion of $F_q(x)$ (at ∞) with respect to $|\,.\,|$,which is isomorphic to $F_q((1/x))$, the field of formal Laurent series in 1/x.

$F_q(x)_\infty^{clos}$ the algebraic closure of $F_q(x)_\infty$.

Ω the completion of $F_q(x)_\infty^{clos}$.

For $m \in N$, let $[m] = x^{q^m} - x$, $[0] = 0$, $L_0 = 1$,
$L_m = [m][m-1]...[1]$,

$$F_0 = 1,\ F_m = [m][m-1]^q...[1]^{q^{m-1}}.$$

It is known (Carlitz[3]) that L_m is the least common multiple of all polynomials of degree m in $F_q[x]$ and F_m is the product of all monic polynomials of degree m in $F_q[x]$. The notions of integrality and divisibility refer to those in the integral domain $F_q[x]$.

We record here auxiliary lemmas which will be used in the proofs of our main theorems. The proofs of these lemmas can be found in Wade.[12,13]

Lemma 1. Every polynomial in $F_q[x]$ divides a linear polynomial $\sum_{j=\ell}^{m} A_j t^{q^j}$, $A_j \in F_q[x]$, $A_\ell \neq 0$, $A_m \neq 0$.

Lemma 2. The expression $\dfrac{[\beta+k_1]^{a_1}...[\beta+k_r]^{a_r}}{[\beta+\ell_1]^{q^j}...[\beta+\ell_s]^{q^j}}$, where $k_1 > ... > k_r > 0$; $\ell_1 \geq ... \geq \ell_s \geq 0$; $j > 0$ are integers independent of β, can be written as a sum of terms that are integral or are of the form

$P\dfrac{[\beta+b_1]^{f_1}...[\beta+b_u]^{f_u}}{[\beta+\ell_1]^{q^j}...[\beta+\ell_v]^{q^j}}$, where $b_1 > ... > b_u$, $0 < f_i \leq q-1$, $v \leq s$, $b_j < \ell_v + j$, and P is a polynomial independent of β. Further, if $q^{t_i} \mid a_i$ ($t_i \geq 0$, i=1, 2, ..., v), then $b_u \geq \min_{i=1,...,r} (\ell_s + j,\ k_i + c_i)$, and the residue mod $[\beta]$ of the integral terms is of smaller degree than the degree of the residue mod $[\beta]$ of the numerator in the original expression.

THE FIRST MAIN THEOREM

Theorem 1. Let e be a positive integer and let $\left(\frac{P_i}{Q_i}\right)$ be a sequence of elements in $F_q(x)$. Assume that

(i) each P_i and Q_i $(\neq 0) \in F_q[x]$,

(ii) $P_i \neq 0$ for infinitely many i,

(iii) $\deg P_i \le (q-1)(i-1)\, q^{i-1} - b_i\, q^i$ when i is sufficiently large and $b_i \to \infty$ $(i \to \infty)$,

(iv) there are only finitly many distinct irreducible factors contained in all Q_i,

(v) there is a non-decreasing sequence of positive integers (d_i) such that $d_i \ge q d_{i-1}$ when i is sufficiently large and $\deg Q_i \le d_i = O(q^i)$ as $i \to \infty$.

Then the series $\alpha_1 := \sum_{k=1}^{\infty} \frac{P_k}{Q_k F_k^e}$, whenever convergent, is transcendental over $F_q(x)$.

Proof. Suppose on the contrary that α_1 is algebraic over $F_q(x)$. Then it is a root of an algebraic equation with coefficients from $F_q(x)$. By Lemma 1, we can put this algebraic equation in the form

$$0=\sum_{j=\ell}^{m} A_j t^{q^j}, \text{ where } A_j \in F_q(x),\ A_\ell \neq 0,\ A_m \neq 0.$$

Direct substitution yields

$$0 = \sum_{j=\ell}^{m} A_j \sum_{k=0}^{\infty}\left(\frac{P_k}{Q_k}\right)^{q^j}\frac{1}{F_k^{eq^j}} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{A_j}{F_k^{eq^j}}\left(\frac{P_k}{Q_k}\right)^{q^j}$$

(where we define $A_j = 0$ if $j > m$ or $j < \ell$)

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{A_j}{F_{k-j}^{eq^j}}\left(\frac{P_{k-j}}{Q_{k-j}}\right)^{q^j}$$ (where we define the terms with negative lower indices to be 0)

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{A_j}{F_{k-j}^{eq^j}}\left(\frac{P_{k-j}}{Q_{k-j}}\right)^{q^j} = \sum_{k=0}^{\infty}\frac{1}{F_k^e}\sum_{i+j=k}\frac{A_j F_k^e}{F_i^{eq^j}}\left(\frac{P_i}{Q_i}\right)^{q^j} = \sum_{k=0}^{\infty}\frac{D_k}{F_k^e}, \qquad (1)$$

where $D_k := \sum_{i+j=k}\frac{A_j F_k^e}{F_i^{eq^j}}\left(\frac{P_i}{Q_i}\right)^{q^j}$.

From (iv), let M be the product of all distinct irreducible factors appeared in all the Q_i and let β be a sufficiently large positive integer to be suitably chosen later. From (1), we get

$$0 = I + R \qquad (2)$$

where $I := M^{d_\beta q^m} F_\beta^e \sum_{k=0}^{\beta}\frac{D_k}{F_k^e}$, $R := M^{d_\beta q^m} F_\beta^e \sum_{k\ge\beta+1}\frac{D_k}{F_k^e}$

We now subdivide the proof into steps.

Step 1. We claim that I is integral.

Each term of I has the form

$$M^{d_\beta q^m} F_\beta^e \frac{D_k}{F_k^e} = \left(\frac{F_\beta}{F_k}\right)^e \sum_{i+j=k} A_j P_i^{q^j}\left(\frac{F_k}{F_i^{q^j}}\right)^e\left(\frac{M^{d_\beta q^m}}{Q_i^{q^j}}\right) (k = 0,1,\ldots,\beta).$$

Since F_β is divisible by F_k, $M^{d_\beta q^m}$ is divisible by $Q_i^{q^j}$, and F_k is divisible by $F_i^{q^j}$, then I is integral.

Step 2. We claim that $\deg R \to -\infty$ $(\beta \to \infty)$.

Each term of R has the form $N := M^{d_\beta q^m}\left(\frac{F_\beta}{F_k}\right)^e D_k$, when $k \ge \beta+1$. Using the definition of D_k, we get $\deg D_k \le \max_{\substack{i+j=k\\ j=\ell,\ldots,m}} \{\deg A_j + q^j(\deg P_i - \deg Q_i) + e(\deg F_k - q^j \deg F_i)\}$.

Let $a := \max(\deg A_j)$. Since $\deg F_i = iq_i$, using (iii) for sufficiently large β, and so also k, we see that deg

$D_k \le a + \max\{\, q^j \deg P_{k-j} + e(\, kq^k - q^j(k-j)q^{k-j}\,)\,\}$

$\le a + \max\{\, q^j(q-1)(k-j-1)q^{k-j-1} - b_{k-j}q^{k-j} + ejq^k\,\}$

$\le a + \max\{\,(1-1/q)k - c_1 b_k' + c_2\,\}q^k$,

where $b_k' := \min_{j=\ell,\ell+1,\ldots,m}(b_{k-j}) \to \infty$ $(\beta \to \infty)$, c_1 and c_2 are positive constants independent of k, β. Thus deg N

$= d_\beta q^m \deg M + \deg D_k + e(\deg F_\beta - \deg F_k)$

$\le d_\beta q^m m_0 + a + \{(1-1/q)k - c_1 b_k' + c_2\}q^k + e(\beta q^\beta - kq^k)$, where $m_0 = \deg M$

$\le d_\beta q^m m_0 + a + q^{\beta+1}\{(1-1/q-e)(\beta+1) - c_1 b_{\beta+1}' + c_2\} + e\,\beta q^\beta$, for sufficiently large $k \ge \beta+1$

$$= a + q^{\beta+1}\left(\frac{d_\beta m_0}{q^{\beta+1-m}} + \beta(1-1/q)(1-e) - c_3 b_{\beta+1}' + c_4\right)$$

where c_3, c_4 are constants independent of k, β $\to -\infty$ $(\beta \to \infty)$, by (iii) and (v).

Consequently, $\deg R \to -\infty$ $(\beta \to \infty)$. From (2) and steps 1 and 2, we get

$$I = R = 0 \text{ when } \beta \text{ is sufficiently large.}$$

Step 3. We claim that $0 = I \equiv M^{d_\beta q^m} D_\beta \left(\bmod \left(\frac{F_\beta}{F_{\beta-1}}\right)^e\right)$ when β is sufficiently large.

From the definition of I, we have

$$I = M^{d_\beta q^m}\left\{D_\beta + D_{\beta-1}\left(\frac{F_\beta}{F_{\beta-1}}\right)^e + \ldots + D_0\left(\frac{F_\beta}{F_0}\right)^e\right\}. \qquad (3)$$

From the proof in step 1, we have $M^{d_\beta q^m} D_k \in F_q[x]$ ($k = 0,1,\ldots,\beta$), F_β is divisible by F_k, and F_β/F_0, F_β/F_1, ..., $F_\beta/F_{\beta-1}$ are divisible by $F_\beta/F_{\beta-1}$. Therefore, (3) gives

$$0 = I \equiv M^{d_\beta q^m} D_\beta \ (\mathrm{mod} \left(\frac{F_\beta}{F_{\beta-1}}\right)^e) \text{ when } \beta \text{ is sufficiently large.}$$

Step 4. We claim that $D_k = 0$ when k is sufficiently large.

When β is sufficiently large, we have

$$\deg\left(\frac{F_\beta}{F_{\beta-1}}\right)^e - \deg M^{d_\beta q^m} D_\beta = e\{\beta q^\beta - (\beta-1)q^{\beta-1}\} - d_\beta q^m m_0 - \deg D_\beta$$

$\geq e\{\beta q^\beta - (\beta-1)q^{\beta-1}\} - c_5 q^{\beta+m} - a - q^\beta\{(1-1/q)\beta - c_1 b_\beta' + c_2\}$, where c_5 is a positive constant from (v) and step 2 $\geq -a + q^\beta\{\beta(e-1)(1-1/q) - c_6 + c_1 b_\beta'\}$, where c_6 is a positive constant

$\to \infty$ as $\beta \to \infty$.

Thus from step 3, since $M \neq 0$, we get $D_\beta = 0$ when β is sufficiently large, and so **there is an index s** > a := max (deg A_j) **such that $D_k = 0$ for all $k \geq s + \ell$.**

Step 5. We claim that

$$0 = \frac{F_s^{eq^\ell}}{F_{s+\ell}^e} M^{d_{s+\ell} q^m} D_{s+\ell} \equiv A_\ell P_s^{q^\ell} \frac{M^{d_{s+\ell} q^m}}{Q_s^{q^\ell}} (\mathrm{mod}[s]^{eq^\ell}).$$

From step 4 and the definition of $D_{s+\ell}$ we have

$$0 = \frac{F_s^{eq^\ell}}{F_{s+\ell}^e} M^{d_{s+\ell} q^m} D_{s+\ell} = F_s^{eq^\ell} M^{d_{s+\ell} q^m} \sum_{i+j=s+\ell} A_j \left(\frac{P_i}{Q_i}\right)^{q^j} \frac{1}{F_i^{eq^j}}.$$

By hypothesis (v), $M^{d_{s+\ell} q^m}$ is divisible by $Q_{s-1}^{q^{\ell+1}}, \ldots, Q_{s+\ell-m}^{q^m}$,

and note that $\frac{F_s^{q^\ell}}{F_{s-1}^{q^{\ell+1}}}, \ldots, \frac{F_s^{q^\ell}}{F_{s+\ell-m}^{q^m}}$ are all congruent to 0 (mod $[s]^{q^\ell}$). The claim thus follows.

Step 6. We claim that for $k \geq s$, and $[k,s] := [k]\,[k-1]^q \ldots [s]^{q^{k-s}}$, we have

$$A_\ell^{\frac{q^{k-s+1}-1}{q-1}} \left(\frac{P_k}{Q_k}\right)^{q^\ell} M^{d_{k+\ell} q^m} \equiv 0 \ (\mathrm{mod}\ [k,s]^{eq^\ell})$$

Proceed by induction on k. The case k = s is step 5. Assume the claim holds for k = s, s+1,..., β-1 (β > s). Now by step 4

$$0 = A_\ell^{\frac{q(q^{\beta-s}-1)}{q-1}} \left(\frac{F_\beta^{q^\ell}}{F_{\beta+\ell}}\right)^e D_{\beta+\ell} M^{d_{\beta+\ell} q^m} =$$

$$A_\ell^{\frac{q(q^{\beta-s}-1)}{q-1}} F_\beta^{eq^\ell} M^{d_{\beta+\ell} q^m} \sum_{i+j=\beta+\ell} A_j \left(\frac{P_i}{Q_i}\right)^{q^j} \frac{1}{F_i^{eq^j}} \quad (j = \ell,\ldots,m)$$

$$:= A_\ell^{\frac{q^{\beta-s+1}-1}{q-1}} \left(\frac{P_\beta}{Q_\beta}\right)^{q^\ell} M^{d_{\beta+\ell} q^m} + T_{\beta-1} + T_{\beta-2} + \ldots + T_{\beta+\ell-m}, \text{ say.}$$

Observe that $\frac{F_\beta^{q^\ell}}{F_{\beta-1}^{q^{\ell+1}}} \equiv 0$ (mod$[\beta]^{q^\ell}$) and from (v), we get $qd_{\beta-1+\ell} \leq d_{\beta+\ell}$ when β is sufficiently large.

$$\text{Therefore, } T_{\beta-1} := A_{\ell+1}\left[A_\ell^{\frac{q^{\beta-s}-1}{q-1}} \left(\frac{P_{\beta-1}}{Q_{\beta-1}}\right)^{q^\ell} M^{d_{\beta-1+\ell} q^m}\right]^q$$

$$M^{(d_{\beta+\ell} - qd_{\beta-1+\ell})q^m} \left(\frac{F_\beta^{q^\ell}}{F_{\beta-1}^{q^{\ell+1}}}\right)^e$$

$\equiv 0$ (mod $[\beta\text{-}1,s]^{eq^{\ell+1}} [\beta]^{eq^\ell}$)

$\equiv 0$ (mod $[\beta,s])^{eq^\ell}$ because $[\beta]\,[\beta\text{-}1,s]^q = [\beta,s]$.

$$\text{Similarly, } T_{\beta-2} := A_{\ell+2}\left[A_\ell^{\frac{q^{\beta-s-1}-1}{q-1}} \left(\frac{P_{\beta-2}}{Q_{\beta-2}}\right)^{q^\ell} M^{d_{\beta-2+\ell} q^m}\right]^{q^2}$$

$$A_\ell^q\, M^{(d_{\beta+\ell} - q^2 d_{\beta-2+\ell})q^m} \left(\frac{F_\beta^{q^\ell}}{F_{\beta-2}^{q^{\ell+2}}}\right)^e$$

$\equiv 0$ (mod ($[\beta\text{-}2, s]^{eq^{\ell+2}} [\beta\text{-}1]^{eq^{\ell+1}} [\beta]^{eq^\ell}$))

$\equiv 0$ (mod $[\beta, s]^{eq^\ell}$)

$$T_{\beta+\ell-m} := A_m\left[A_\ell^{\frac{q^{\beta+\ell-m-s+1}-1}{q-1}} \left(\frac{P_{\beta+\ell-m}}{Q_{\beta+\ell-m}}\right)^{q^\ell} M^{d_{\beta-m+2\ell} eq^L}\right]^{q^{m-\ell}}$$

$$A_\ell^{\frac{q(q^{m-\ell-1}-1)}{q-1}}\, M^{(d_{\beta+\ell} - q^{m-\ell} d_{\beta-m+2\ell})q^m} \times \left(\frac{F_\beta^{q^\ell}}{F_{\beta+\ell-m}}\right)^e$$

$\equiv 0$ (mod $[\beta+\ell\text{-}m,s]^{eq^m} [\beta+\ell\text{-}m+1]^{eq^{m-1}} \ldots$

$[\beta-1])^{eq^{\ell+1}} [\beta]^{eq^\ell}$) (by (v))

$\equiv 0$ (mod $[\beta,s]^{eq^\ell}$)

Step 7. Finally, we now derive a desired contradiction. From (ii) and step 6, we see that for infinitely many and sufficiently large k,

q^{ℓ} deg $P_k \geq eq^{\ell}$ deg [k,s]- $\frac{q^{k-s+1}-1}{q-1}$ deg A_{ℓ} + q^{ℓ} deg Q_k - $d_{k+\ell}\, q^m$ deg M

$\geq e\,(k - s + 1)q^{k+\ell} - q^{k-s+1}a - c_7\, q^{k+\ell}$, where c_7 is a positive constant independent of k.

Thus when k is sufficiently large,

deg $P_k \geq e\,(k - s + 1)q^k - c_8\, q^{k+\ell}$, where c_8 is a positive constant independent of k

$> (q-1)(k-1)q^{k-1} - b_k q^k$.

This contradicts (iii) unless $P_k = 0$ for all sufficiently large k, yet this only other possibility in turn contradicts (ii). Hence, α_1 is not algebraic over $F_q(x)$.

THE SECOND MAIN THEOREM

Theorem 2. Let e be a nonnegative integer and let $\left(\frac{P_i}{Q_i}\right)$ be a sequence of elements in $F_q(x)$. Assume that

(i) each P_i and Q_i ($\neq 0$) $\in F_q[x]$,

(ii) $P_i \not\equiv 0$ (mod [i]) for infinitely many i ,

(iii) deg $P_i = \begin{cases} o(q^i) & \text{if } q\neq 2 \\ o(q^{i/2}) & \text{if } q=2 \end{cases}$ $(i \to \infty)$,

(iv) there are only finitely many distinct irreducible factors contained in all Q_i ,

(v) there is a nondecreasing sequence of nonnegative integers (d_i) such that

$$\deg Q_i \leq d_i = \begin{cases} o(q^i) & \text{if } q\neq 2 \\ o(q^{i/2}) & \text{if } q=2 \end{cases} \quad (i \to \infty).$$

Then the series $\alpha_2 := \sum_{k=1}^{\infty} \frac{P_k}{Q_k\, [k]_k^{q^e}}$, whenever convergent, is transcendental over $F_q(x)$.

Proof. Assume to the contrary that α_2 is algebraic over $F_q(x)$. Then by Lemma 1, α_2 is a root of an algebraic equation of the shape $0 = \sum_{j=\ell}^{m} A_j t^{q^j}$, where $A_j \in F_q[x]$, $A_\ell \neq 0$, $A_m \neq 0$. Direct substitution yields

$$0 = \sum_{j=a}^{m} A_j \sum_{k=1}^{\infty} \left(\frac{P_k}{Q_k}\right)^{q^j} \frac{1}{[k]^{q^{j+e}}}.$$

Multiplying this equation by $M^{d_{2\beta+m} q^m} \frac{L_{2\beta}^{q^{m+e}}}{L_\beta}$, where, by (iv), M is the product of all distinct irreducible factors appeared in all the Q_i , and β is an integer to be chosen sufficiently large, and separating terms, we get an equation of the shape

$$0 = T_1 + T_2 + T_3 \qquad (1)$$

where

$$T_1 := M^{d_{2\beta+m} q^m} \frac{L_{2\beta}^{q^{m+e}}}{L_\beta} \sum_{j=\ell}^{m} A_j \sum_{k=1}^{2\beta} \left(\frac{P_k}{Q_k}\right)^{q^j} \frac{1}{[k]^{q^{j+e}}},$$

$$T_2 := M^{d_{2\beta+m} q^m} \frac{L_{2\beta}^{q^{m+e}}}{L_\beta} \sum_{j=\ell}^{m} A_j \sum_{s=1}^{m-j} \left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j} \frac{1}{[2\beta+s]^{q^{j+e}}},$$

$$T_3 := M^{d_{2\beta+m} q^m} \frac{L_{2\beta}^{q^{m+e}}}{L_\beta} \sum_{j=\ell}^{m} A_j \sum_{k=2\beta+m-j+1}^{\infty} \left(\frac{P_k}{Q_k}\right)^{q^j} \frac{1}{[k]^{q^{j+e}}}.$$

We split T_2 by noting that $\left(\frac{L_{2\beta}^{q^m}}{[2\beta+s]^{q^j}}\right)^{q^e} = [2\beta+s]^{(q^{m-s}-q^j)q^e} L_{2\beta-1}^{q^{m+e}}$

$$+ \sum_{k=0}^{m-s-j-1} (-1)^{k+1} \left(\prod_{i=0}^{k} [s+i]^{q^{m-s-i+e}}\right) [2\beta+s]^{q^{m-s-k-1+e}-q^{j+e}} L_{2\beta-k-2}^{q^{m+e}}$$

$$+ \frac{(-1)^{m-s-j-1} \prod_{i=0}^{m-s-j} [s+i]^{q^{m-s-i+e}} L_{2\beta-m+s+j-1}^{q^{m+e}}}{[2\beta+s]^{q^{j+e}}}.$$

Now put $I = T_1 + T_{21}$, where $T_2 = T_{21} + T_{22}$,

$$T_{21} := \sum_{j=\ell}^{m} A_j \sum_{s=1}^{m-j} [2\beta+s]^{q^{m-s+e}-q^{j+e}} \frac{L_{2\beta-1}^{q^{m+e}}}{L_\beta} M^{d_{2\beta+m} q^m} \left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j}$$

$$+ \sum_{j=\ell}^{m} A_j \sum_{s=1}^{m-j} \sum_{k=0}^{m-s-j-1} (-1)^{k+1} \left(\prod_{i=0}^{k} [s+i]^{q^{m-s-i+e}}\right)$$

$$[2\beta+s]^{q^{m-s-k-1+e}-q^{j+e}} \frac{L_{2\beta-k-2}^{q^{m+e}}}{L_\beta} M^{d_{2\beta+m} q^m} \left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j}$$

$$T_{22} := \sum_{j=\ell}^{m} A_j \sum_{s=1}^{m-j} (-1)^{m-s-j-1} \left(\prod_{i=0}^{m-s-j} [s+i]^{q^{m-s-i+e}}\right)$$

$$\frac{L_{2\beta-m+s+j-1}^{q^{m+e}}}{L_\beta} \left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j} \frac{M^{d_{2\beta+m} q^m}}{[2\beta+s]^{q^{j+e}}}.$$

Step 1. We claim that I is integral when β is sufficiently large.

For k=1,…,2β, since $\frac{L_{2\beta}^{q^{m+e}}}{L_\beta} \frac{1}{[k]^{q^{e+j}}}$ and $\left(\frac{P_k}{Q_k}\right)^{q^j} M^{d_{2\beta+m} q^m}$ are integral, then T_1 is integral.

Since $L_\beta \mid L_{2\beta-1}$ and $Q^{q^j}_{2\beta+s} \mid M^{d_{2\beta+m}q^m}$, then the first sum in T_{21} is integral, while the second sum is also integral because, for sufficiently large β, $L_\beta \mid L_{2\beta-k-2}$. Thus $I = T_1 + T_{21}$ is integral when β is sufficiently large.

Next, consider $Q := T_{22} + T_3$.

Step 2. We claim that $\deg Q \to -\infty \ (\beta\to\infty)$.

Since $\deg\left(\dfrac{L^{q^{m+e}}_{2\beta-m+s+j-1}}{L_\beta[2\beta+s]^{q^{j+e}}}\right) = \left(\dfrac{q^{2\beta-m+s+j-1}-q}{q-1}\right)q^{m+e} - \dfrac{q^{\beta+1}-q}{q-1} - q^{2\beta+s+j+e}$

$\le -c_1 q^{2\beta}$, where c_1 is a positive constant independent of β, and

$$\deg\left\{\left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j} M^{d_{2\beta+m}q^m}\right\} \le q^j \deg P_{2\beta+s} + d_{2\beta}q^m \deg M \le o(q^{2\beta}),$$

by (iii) and (v), then $\deg T_{22} \to -\infty \ (\beta\to\infty)$. On the other hand, when $k \ge 2\beta+1$ and β sufficiently large, we similarly have

$$\deg\left\{\left(\frac{P_k}{Q_k}\right)^{q^j} M^{d_{2\beta+m}q^m}\right\} \le o(q^k), \quad \deg\left(\frac{L^{q^{m+e}}_{2\beta}}{L_\beta[k]^{q^{j+e}}}\right) \le -c_2 q^k,$$

where c_2 is a positive constant independent of β, so $\deg T_3 \to -\infty \ (\beta\to\infty)$. Thus $\deg Q \to -\infty \ (\beta\to\infty)$.

From (1), we get $I + Q = 0$, and by steps 1 and 2, we deduce that $I = 0$, $Q = 0$.

Step 3. We now show that $I \ne 0$; this gives a desired contradiction.

Write $I = T_1 + T_{21} = T_1 + E_2 + E_3$, where $T_{21} = E_2 + E_3$,

$$E_2 := \sum_{j=\ell}^{m} A_j \sum_{s=1}^{m-j} [2\beta+s]^{q^{m-s+e}-q^{j+e}} \frac{L^{q^{m+e}}_{2\beta-1}}{L_\beta} M^{d_{2\beta+m}q^m} \left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j},$$

$$E_3 := \sum_{j=\ell}^{m} A_j \sum_{s=1}^{m-j} \sum_{k=0}^{m-s-j-1} (-1)^{k+1} \left(\prod_{i=0}^{k} [s+i]^{q^{m-s-i+e}}\right) [2\beta+s]^{q^{m-s-k-1+e}-q^{j+e}} \frac{L^{q^{m+e}}_{2\beta-k-2}}{L_\beta} M^{d_{2\beta+m}q^m} \left(\frac{P_{2\beta+s}}{Q_{2\beta+s}}\right)^{q^j}.$$

We know from step 1 that E_2 is integral and since $\dfrac{L_{2\beta-1}}{L_\beta} \equiv 0 \pmod{[2\beta-m]}$, so is E_2. Similarly, for E_3. Now from step 1, T_1 is integral and the terms in T_1 contain the factors

$$\frac{L^{q^{m+e}}_{2\beta}}{L_\beta}\frac{1}{[k]^{q^{j+e}}} \equiv \begin{cases} 0 & \pmod{[2\beta-m]} \quad \text{if } k \ne 2\beta-m \\ 0 & \pmod{[2\beta-m]} \quad \text{if } k = 2\beta-m, j<m \\ \dfrac{L^{q^{m+e}}_{2\beta}}{L_\beta}\dfrac{1}{[2\beta-m]^{q^{m+e}}} & \pmod{[2\beta-m]} \quad \text{if } k = 2\beta-m, j=m \end{cases}$$

Thus $I \equiv \dfrac{L^{q^{m+e}}_{2\beta}}{L_\beta}\dfrac{A_m}{[2\beta-m]^{q^{m+e}}}\left(\dfrac{P_{2\beta-m}}{Q_{2\beta-m}}\right)^{q^m} M^{d_{2\beta+m}q^m} \pmod{[2\beta-m]}$

$$\equiv A_m \left(\frac{P_{2\beta-m}}{Q_{2\beta-m}}\right)^{q^m} M^{d_{2\beta+m}q^m} \frac{L^{q^{m+e}}_{2\beta-m-1}}{L_\beta} L_m^{q^{2\beta+e}} \pmod{[2\beta-m]}$$

$\not\equiv 0 \pmod{[2\beta-m]}$, by (ii) and for β sufficiently large.

This shows that $I \ne 0$, and the proof is complete.

The Third Main Theorem

Theorem 3. Let e be a positive integer and let $\left(\dfrac{P_i}{Q_i}\right)$ be a sequence of elements in $F_q(x)$. Assume that

(i) each P_i and $Q_i \ (\ne 0) \in F_q[x]$,

(ii) $P_i \ne 0$ for infinitely many i,

(iii) $\deg P_i = o(q^i) \ (i\to\infty)$

(iv) there are only finitly many distinct irreducible factors contained in all Q_i,

(v) there is a non-decreasing sequence of non-negative integers (d_i) such that $\deg Q_i \le d_i = o(q^i) \ (i\to\infty)$

(vi) $\deg\left(\dfrac{P_i}{Q_i}\right) \ge \deg\left(\dfrac{P_{i+1}}{Q_{i+1}}\right)$ for sufficiently large i.

Then the series $\alpha_3 := \sum_{k=1}^{\infty} \dfrac{P_k}{Q_k L_k^e}$, whenever convergent, is transcendental over $F_q(x)$.

Proof. Asuume on the contrary that α_3 is algebraic over $F_q(x)$. Then by Lemma 1, α_3 is a root of an algebraic equation of the shape

$$0 = \sum_{j=0}^{m} A_j t^{q^j}, \text{ where } A_j \in F_q[x], A_m \ne 0 \tag{1}$$

Without loss of generality, take $m > e+3$. Let

$J_{\beta,r} := \prod_{i=1}^{r} \prod_{k=(i-1)m+1}^{im} [\beta+k]^{eq^{m-i}} \ (1\le r\le m)$, $K_\beta := L_\beta^{eq^m} J_{\beta,m}$,

where β is a positive integer to be suitably chosen, and let, by (iv), M be the product of all distinct irreducible factors appeared in all the Q_i. Substituting for α_3 into (1) and multiplying by $M^{d_{\beta+m^2} q^m} K_\beta$, we get $0 = I_1 + Q_1 + R_1$, where $I_1 := M^{d_{\beta+m^2} q^m} K_\beta \sum_{j=0}^{m} A_j \sum_{k=0}^{\beta+(m-j)m} \left(\frac{P_k}{Q_k}\right)^{q^j} \frac{1}{L_k^{eq^j}}$,

$$Q_1 := M^{d_{\beta+m^2} q^m} K_\beta \sum_{j=0}^{m} A_j \sum_{k=m^2-j+1}^{\infty} \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^j} \frac{1}{L_{\beta+k}^{eq^m}},$$

$$R_1 := M^{d_{\beta+m^2} q^m} K_\beta \sum_{j=0}^{m} A_j \sum_{k=(m-j)m+1}^{m^2-j} \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^j} \frac{1}{L_{\beta+k}^{eq^j}};$$ note that in R_1, the sum over j starts from j = 1.

Step 1. We claim that I_1 is integral.

Since $Q_k^{q^j} \mid M^{d_{\beta+m^2} q^m}$, and $L_k^{eq^j} \mid K_\beta (k = 0,..., \beta+(m-j)m ; j = 0,...,m)$, then I_1 is integral.

Step 2. We now analyze R_1.
Consider each term in the sum of R_1. The term with j = m is

$$T_m := M^{d_{\beta+m^2} q^m} K_\beta A_m. \sum_{k=1}^{m^2-m} \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^m} \frac{1}{L_{\beta+k}^{eq^m}}$$

Let $H(m,t) := \frac{K_\beta}{L_{\beta+t}^{eq^m}} = \frac{\prod_{i=1}^{m} \prod_{k=(i-1)m+1}^{im} [\beta+k]^{eq^{m-i}}}{[\beta+1]^{eq^m} \ldots [\beta+t]^{eq^m}}$ (t=1,..., m^2-m). Then each term of T_m can be written as

$$M^{d_{\beta+m^2} q^m} A_m \left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)^{q^m} H(m,t) \ (t = 1,...,m^2-m). \quad (2)$$

The term with j = m-1 is $T_{m-1} := M^{d_{\beta+m^2} q^m} K_\beta A_{m-1} \sum_{k=m+1}^{m^2-m+1} \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^{m-1}} \frac{1}{L_{\beta+k}^{eq^{m-1}}}$.

Since $\frac{K_\beta}{L_{\beta+m+t}^{eq^{m-1}}} = \frac{L_\beta^{eq^m - eq^{m-1}} \prod_{i=2}^{m} \prod_{k=(i-1)m+1}^{im} [\beta+k]^{eq^{m-i}}}{[\beta+m+1]^{eq^{m-1}} \ldots [\beta+m+t]^{eq^{m-1}}}$ (t=1,..., m^2-2m+1), then each term of T_{m-1} is of the form

$$M^{d_{\beta+m^2} q^m} A_{m-1} \left(\frac{P_{\beta+m+t}}{Q_{\beta+m+t}}\right)^{q^{m-1}} H(m-1,t)\, L_\beta^{eq^m - eq^{m-1}}, \quad (3)$$

where $H(m-1,t) := \frac{\prod_{i=2}^{m} \prod_{k=(i-1)m+1}^{im} [\beta+k]^{eq^{m-i}}}{[\beta+m+1]^{eq^{m-1}} \ldots [\beta+m+t]^{eq^{m-1}}}$

(t= 1,...,m^2-2m+1).
The terms with $1 \le j \le m-2$ are of the form $T_{1\le j\le m-2} := M^{d_{\beta+m^2} q^m} L_\beta^{eq^m} J_{\beta,m} A_j \sum_{k=(m-j)m+1}^{m^2-j} \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^m} \frac{1}{L_{\beta+k}^{eq^m}}$.

Since $\frac{J_{\beta,m}}{L_{\beta+(m-j)m+t}^{eq^j}} = \frac{J_{\beta,m-j-1} \prod_{i=m-j+1}^{m} \prod_{k=(i-1)m+1}^{im} [\beta+k]^{eq^{m-i}}}{\{[\beta+(m-j)m+t] \ldots [\beta+(m-j)m+1]\}^{eq^j} L_{\beta+(m-j-1)m}^{eq^j}}$, t = 1,...,mj-j, then each term of $T_{1\le j\le m-2}$ is of the form

$$M^{d_{\beta+m^2} q^m} A_j \left(\frac{P_{\beta+(m-j)m+t}}{Q_{\beta+(m-j)m+t}}\right)^{q^{m-1}} \frac{L_\beta^{eq^m}}{L_{\beta+(m-j-1)m}^{eq^j}} H(j,t)\, J_{\beta,m-j-1}, \quad (4)$$

$$\text{where} H(j,t) := \frac{\prod_{i=m-j+1}^{m} \prod_{k=(i-1)m+1}^{im} [\beta+k]^{eq^{m-i}}}{\{[\beta+(m-j)m+t] \ldots [\beta+(m-j)m+1]\}^{eq^j}}$$

$$= \frac{\{\prod_{k=(m-j)m+1}^{(m-j+1)m} [\beta+k]^{eq^{j-1}}\} \ldots \{\prod_{k=(m-1)m+1}^{m^2} [\beta+k]^{e}\}}{\{[\beta+(m-j)m+t] \ldots [\beta+(m-j)m+1]\}^{eq^j}}$$

(t = 1,...,mj-j). From Lemma 2, we see that each of H(m,t), H(m-1,t) and H(j,t) can be written as sum of integral terms and of non-integral terms with the properties given in the lemma. Thus each term of T_m, of T_{m-1} and of $T_{1\le j\le m-2}$ can be expressed as sum of integral terms and of non-integral terms with the properties mentioned in Lemma 2. Let

$I := I_1$ + sum of integral terms from R_1, $Q := Q_1$ + sum of non-integral terms from R_1.
Thus I + Q = 0.

Step 3. We cliam that deg $Q \to -\infty$ ($\beta \to \infty$).
Since $Q := Q_1$ + sum of non-integral terms from R_1, let N_1 be a term in Q_1. Then for $k \ge m^2 - j + 1$,

$$N_1 := M^{d_{\beta+m^2} q^m} K_\beta A_j \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^j} \frac{1}{L_{\beta+k}^{eq^j}} = M^{d_{\beta+m^2} q^m} \left(\frac{P_{\beta+k}}{Q_{\beta+k}}\right)^{q^j} A_j J_{\beta,m} \frac{L_\beta^{eq^m}}{L_{\beta+k}^{eq^j}}.$$

$$\deg N_1 \le \deg L_\beta^{eq^m} + \deg J_{\beta,m} + \deg A_j - \deg L_{\beta+k}^{eq^j} + d_{\beta+m^2} q^m \deg M + q^j \deg P_{\beta+k}$$

$$< eq^m \frac{q^{\beta+1}-q}{q-1} + \frac{e}{q-1}\left(q^{\beta+m(m-1)+m+1} + q^{\beta+(m-1)^2+m+1}\right)$$

$$+ a - \frac{eq^j}{q-1}\left(q^{\beta+k+1}-q\right) + d_{\beta+m^2} q^m m_0 + q^j \deg P_{\beta+k}$$,where

$a := \max_{0\le j\le m} \deg A_j$, $m_0 := \deg M$

$$\le \frac{eq^{\beta+k+j+1}}{q-1}\{-1 + c_1 q^{m^2-k-j-1} + \frac{c_2 o(q^{\beta+m^2})}{q^{\beta+k+j+1}} + \frac{1}{q^{\beta+k+1}} o(q^{\beta+k})\} + a,$$

where c_1 and c_2 are positive constants independent of β,

$\to -\infty (\beta \to \infty)$

Since $R_1 = T_m + T_{m-1} + T_{1\le j\le m-2}$, consider an arbitrary term in the sum of non-integral terms in R_1,by Lemma 2, H(j,t), j = 0,...,m, can be written as a sum of integral terms and non-integral terms of the form

$$NIH(j,t) := \frac{P[\beta+m(m-j)+g+j-1]^{\ell_g}\dots[\beta+m(m-j)+j]^{\ell_1}}{[\beta+m(m-j)+t]^{eq^j}\dots[\beta+m(m-j)+g+1]^{eq^j}[\beta+m(m-j)+g]^{\delta q^j}},$$

where $1\le t\le m^2-j, 1\le g\le t, 1\le \delta\le e, 0\le \ell_i \le q-1$ (i = 1,2,...,g), and P is independent of β.

Let $b := \max_P \deg P$; this value is independent of β.

Note that deg NIH(j,t) is maximal when each $\ell_i = q-1$, $g = t, \delta = 1$, and so $\deg NIH(j,t) \le b+(q-1)(q^{\beta+m(m-j)+t+j-1} + \dots + q^{\beta+m(m-j)+j}) - q^{\beta+m(m-j)+t+j} = b - q^{\beta+m(m-j)+j}$.

Therefore, by (2), (3) and (4)

$$\deg(\text{ non-integral term in } T_m) \le d_{\beta+m^2} + q^m m_0 + a + q^m(\deg P_{\beta+t} - \deg Q_{\beta+t}) + \deg NIH(m,t)$$

$$= d_{\beta+m^2} q^m m_0 + a + q^m o(q^{\beta+t}) + b - q^{\beta+m}$$

$$= a + b - q^{\beta+m}\left[1 - \frac{d_{\beta+m^2} m_0}{q^\beta} + \frac{q^t o(q^{\beta+t})}{q^{\beta+t}}\right] \to -\infty (\beta\to\infty)$$

$(t=1,2,\dots,m^2-m)$

$$\deg(\text{non-integral term in } T_{m-1}) \le d_{\beta+m^2} q^m m_0 + a + \deg NIH(m-1,t) + (eq^m - eq^{m-1})\deg L_\beta + q^{m-1} \deg P_{\beta+m+t}$$

$$= d_{\beta+m^2} q^m m_0 + a + b - q^{\beta+2m-1} + eq^{m-1}(q-1)\frac{q^{\beta+1}-q}{q-1} + q^{m-1} o(q^{\beta+m+t})$$

$$= a + b - q^{\beta+2m-1}\left[1 - \frac{d_{\beta+m^2} m_0}{q^{\beta+m-1}} - \frac{e(q^\beta-1)}{q^{\beta+m-1}} - \frac{q^t o(q^{\beta+m+t})}{q^{\beta+m+t}}\right]$$

$\to -\infty (\beta\to\infty)$

$\deg(\text{non-integral term in } T_{1\le j\le m-2})$

$$\le d_{\beta+m^2} q^m m_0 + a + \deg NIH(j,t) + eq^m \deg L_\beta + \deg J_{\beta,m-j-1} - eq^j \deg L_{\beta+m(m-j-1)} + q^j \deg P_{\beta+m(m-j)+t}$$

$$< d_{\beta+m^2} q^m m_0 + a + b - q^{\beta+m(m-j)+j} + eq^m \frac{q^{\beta+1}-q}{q-1} + \frac{e}{q-1}\left(q^{\beta+(m-j-1)(m-1)+m+1} + q^{\beta+(m-j-2)(m-1)+m+1}\right)$$

$$-eq^j \frac{q^{\beta+(m-j-1)m+1}-q}{q-1} + q^j o(q^{\beta+(m-j)m+t})$$

$$< a + b - q^{\beta+(m-j)m+j}$$

$$\left[1 - \frac{d_{\beta+m^2} m_0}{q^{\beta+(m-j)m+j-m}} - \frac{e}{q^{(m-j)m+j-m-1}} - \frac{e}{q^{m-2}} - \frac{e}{q^{2m-3}} + \frac{e}{q^m} - \frac{q^t o(q^{\beta+(m-j)m+t})}{q^{\beta+(m-j)m+t}}\right]$$

$\to -\infty (\beta\to\infty)$

Thus, deg (each term in the sum of non-integral terms in R_1)$\to -\infty$,and so deg $Q \to -\infty (\beta\to\infty)$

From I + Q = 0 ,steps 1 and 3, we conclude that I = Q = 0.

Step 4. We derive a desired contradiction by showing that $I \ne 0$.

Since $I = I_1$ + sum of integral terms from R_1, then consider

$$I_1 := M^{d_{\beta+m^2} q^m} L_\beta^{eq^m} J_{\beta,m} \sum_{j=0}^{m} A_j \sum_{k=0}^{\beta+(m-j)m} \left(\frac{P_k}{Q_k}\right)^{q^j} \frac{1}{L_k^{eq^j}}.$$

Observe that

$$\frac{L_\beta^{eq^m} J_{\beta,m}}{L_k^{eq^j}} \equiv \begin{cases} 0 \pmod{[\beta]} & \text{for } j=0,\dots,m-1; k=0,\dots,\beta+(m-j)m \text{ and for } j=m; k=0,\dots,\beta-1 \\ J_{\beta,m} \pmod{[\beta]} & \text{for } j=m; k=\beta \end{cases}$$

Then using $[\beta + k] \equiv [k] \pmod{[\beta]}$, we get

$$I_1 \equiv M^{d_{\beta+m^2} q^m} A_m \left(\frac{P_\beta}{Q_\beta}\right)^{q^m} \prod_{i=1}^{m} \prod_{k=(i-1)m+1}^{im} [k]^{eq^{m-i}} \pmod{[\beta]}.$$

Now consider integral terms from $R_1 = T_m + T_{m-1} + T_{1\le j\le m-2}$.

The integral terms from T_m are of the form

$$M^{d_{\beta+m^2} q^m} A_m H(m,t) \left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)^{q^m} \quad (t = 1,2,\dots,m^2-m)$$

The integral terms from T_{m-1} are of the form

$$M^{d_{\beta+m^2} q^m} A_{m-1} H(m-1,t) L_\beta^{eq^m - eq^{m-1}} \left(\frac{P_{\beta+m+t}}{Q_{\beta+m+t}}\right)^{q^{m-1}} \equiv 0$$

$\pmod{[\beta]}$. $(t = 1,2,\dots,m^2-m+1)$.

The integral terms from $T_{1\le j\le m-2}$ are of the form

$$M^{d_{\beta+m^2}q^m} A_j L_\beta^{eq^m} \frac{J_{\beta,m-j-1}}{L^{eq^j}_{\beta+(m-j-1)m}} H(j,t)\left(\frac{P_{\beta+(m-j)m+t}}{Q_{\beta+(m-j)m+t}}\right)^{q^j} \equiv 0$$

(mod [β]). (t = 1,2,...,mj-j).

Thus integral terms from R_1 ≡ integral terms of the form $M^{d_{\beta+m^2}q^m} A_m H(m,t)\left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)^{q^m}$ (mod [β]).

By (ii) and (iii), we have P_β and $P_{\beta+t}$ are both $\not\equiv 0$ (mod [β]) for β sufficiently large. This together with (iv) and (v) imply that the residue mod [β] of $M^{d_{\beta+m^2}q^m} A_m\left(\frac{P_\beta}{Q_\beta}\right)^{q^m}$ in I_1 and of $M^{d_{\beta+m^2}q^m} A_m\left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)^{q^m}$ are both $\not\equiv 0$ (mod [β]) for β sufficiently large. Thus deg (residue mod [β] of integral terms from R_1)

= deg (residue mod [β] of the integral terms of the form $M^{d_{\beta+m^2}q^m} A_m H(m,t)\left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)^{q^m}$)

$< \deg(M^{d_{\beta+m^2}q^m} A_m) + q^m \deg\left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)$ + deg (numerator of H(m,t) mod [β]) (by Lemma 2)

$= \deg (M^{d_{\beta+m^2}q^m} A_m)+q^m\deg\left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)+\deg \left(\prod_{i=1}^{m} \prod_{k=(i-1)m+1}^{im}[\beta+k]^{eq^{m-1}}\right.$ mod [β])

$= \deg (M^{d_{\beta+m^2}q^m} A_m) + q^m\deg\left(\frac{P_{\beta+t}}{Q_{\beta+t}}\right)+\deg \left(\prod_{i=1}^{m} \prod_{k=(i-1)m+1}^{im}[k]^{eq^{m-1}}\right.$ mod [β])

$\le \deg(M^{d_{\beta+m^2}q^m} A_m)+q^m\deg\left(\frac{P_\beta}{Q_\beta}\right)^{q^m} +\deg \left(\prod_{i=1}^{m} \prod_{k=(i-1)m+1}^{im}[k]^{eq^{m-1}}\right.$ mod [β]) (by (vi))

= deg (residue mod [β] of I_1)

Hence, for sufficiently large, I $\not\equiv 0$ (mod [β]), which yields I ≠ 0.

The fourth main theorem

Theorem 4. Let $G \in F_q[x]$, deg G > 0, $\gamma \in N$, $\gamma > 1$, γ not a power of p. Suppose that $\left(\frac{P_i}{Q_i}\right)$ is a sequence of elements in $F_q(x)$ with the following **properties**

(i) each P_i ,Q_i (≠ 0) ∈ $F_q[x]$

(ii) $P_i \neq 0$ for infinitely many i

(iii) deg $P_i = o(\gamma^i)$ (i→∞)

(iv) there are only finitely many distinct irreducible factors contained in all Q_i

(v) there is a nondecreasing sequence of non-negative integers (d_i) such that deg $Q_i \le d_i = o(\gamma^i)$ ($i \to \infty$).

Then $\alpha_4 := \sum_{i=0}^{\infty} \frac{P_i}{Q_i}\frac{1}{G^{\gamma^i}}$ whenever is convergent, represents an element transcendental over $F_q(x)$.

Proof. Let,by (iv), M be the product of all distinct irreducible factors of all Q_i , β be a positive integer to be suitably chosen later, and let (k_0 , k_1 , ... , k_m) be a decreasing sequence of integers defined by

$p^{m-j} / \gamma < \gamma^{k_j} \le p^{m-j}$, i.e. (m-j) $\log_\gamma p - 1 < k_j \le$ (m - j) $\log_\gamma p$.

Note that when j = m, we have $1/\gamma < \gamma^{k_m} \le 1$ and so $k_m = 0$. When 0 ≤ j < m, we have $\gamma^{k_j} \neq p^{m-j}$ because γ is not a power of prime p which yields $p^{m-j} / \gamma < \gamma^{k_j} < p^{m-j}$. Assume on the contrary that α_4 when its represented series is convergent, is algebraic over $F_q(x)$. Since $F_q(x)$ is an algebraic extension of $F_p(x)$, then α_4 is algebraic over $F_p(x)$,and so is a root of a linear equation of the form $\sum_{j=0}^{m} A_j t^{p^j} = 0$, $A_m \neq 0$, $A_j \in F_p[x]$, Substituting for t by α_4 and multiplying by $G^{\gamma^\beta p^m} M^{d_{\beta+k_0}p^m}$, we get

$$0 = I + Q \text{ , where} I := G^{\gamma^\beta p^m} M^{d_{\beta+k_0}p^m} \sum_{j=0}^{m} A_j \sum_{k=0}^{\beta+k_j}\left(\frac{P_k}{Q_k}\right)^{p^j}\frac{1}{G^{\gamma^k p^j}},$$

$$Q := G^{\gamma^\beta p^m} M^{d_{\beta+k_0}p^m} \sum_{j=0}^{m} A_j \sum_{k\ge\beta+k_j+1}\left(\frac{P_k}{Q_k}\right)^{p^j}\frac{1}{G^{\gamma^k p^j}}.$$

Step 1. We claim that I is integral.

This is trivial because of (iv), (v) and the definition of (k_j).

Let $\mu := \min_{j=0,1,\ldots,m-1} (\gamma^\beta p^m - \gamma^{\beta+k_j} p^j , \gamma^\beta p^m - \gamma^{\beta-1} p^m) = O(\gamma^\beta)$ by the definition of (k_j). Therefore,

$$I = G^{\gamma^\beta p^m} M^{d_{\beta+k_0}p^m} \sum_{j=0}^{m-1} A_j \sum_{k=0}^{\beta+k_j}\left(\frac{P_k}{Q_k}\right)^{p^j}\frac{1}{G^{\gamma^k p^j}}$$

$$+M^{d_{\beta+k_0}p^m} A_m \sum_{k=0}^{\beta-1} G^{\gamma^\beta p^m-\gamma^k p^m} + M^{d_{\beta+k_0}p^m} A_m \left(\frac{P_\beta}{Q_\beta}\right)^{p^m}$$

$$\equiv M^{d_{\beta+k_0}p^m} A_m \left(\frac{P_\beta}{Q_\beta}\right)^{p^m} \not\equiv 0 \pmod{G^\mu}$$, for β sufficiently large, by (iii) and (iv).

Step 2. We claim that $\deg Q \to -\infty$ ($\beta \to \infty$).
Let N be an arbitrary term of Q.

Then $N := G^{\gamma^\beta p^m} M^{d_{\beta+k_0}p^m} A_j \left(\frac{P_k}{Q_k}\right)^{p^j} \frac{1}{G^{\gamma^k p^j}}$ (j=0,..., m; $k \geq \beta + k_j + 1$).

$\deg N = (\gamma^\beta p^m - \gamma^k p^j)\deg G + \deg A_j + p^m d_{\beta+k_0} \deg M + p^j(\deg P_k - \deg Q_k)$

$\leq \gamma^\beta p^j (p^{m-j} - \gamma^{k-\beta})g + a + p^m o(\gamma^\beta)m_0 + p^j o(\gamma^k)$,

where $g = \deg G$, $a = \deg A_j$, $m_0 = \deg M$

$$\leq a + \gamma^\beta p^j \left[\frac{p^{m-j}m_0 o(\gamma^\beta)}{\gamma^\beta} + p^{m-j}g - \frac{g\gamma^k}{\gamma^\beta} + \frac{o(\gamma^k)}{\gamma^\beta}\right]$$

$\to -\infty$ ($\beta \to \infty$), because $k - \beta \geq k_j + 1$, $k \to \infty$ faster than β.

Thus, $\deg Q \to -\infty$ ($\beta \to \infty$). Since $0 = I + Q$, then Claims 1 and 2 together imply that $I = Q = 0$ which contradicts the fact that $I \not\equiv 0 \pmod{G^\mu}$ for sufficiently large β. The contradiction proves the theorem.

THE FIFTH MAIN THEOREM

Theorem 5. Let $G \in F_q[x]$, $\deg G > 0$, $\gamma \in N$, $\gamma > 1$. Suppose that $\left(\frac{P_i}{Q_i}\right)$ is a sequence of elements in $\mathbf{F}_q(\mathbf{x})$ with the following properties

(i) each $P_i, Q_i (\neq 0) \in F_q[x]$

(ii) $P_i \neq 0$ for infinitely many i which are not congruent to 0 (mod p)

(iii) $\deg P_i = o(i^{\gamma-1})$ ($i \to \infty$).

(iv) there are only finitely many distinct irreducible factors contained in all Q_i

(v) there is a nondecreasing sequence of non-negative integers (d_i) such that $\deg Q_i \leq d_i = o(i^{\gamma-1})$ ($i \to \infty$).

Then $\alpha_5 := \sum_{i=0}^{\infty} \frac{P_i}{Q_i} \frac{1}{G^{i^\gamma}}$ whenever convergent, represents an element transcendental over $F_q(x)$.

Proof. Let, by (iv), M be the product of all distinct irreducible factors of all Q_i, β be a positive integer not divisible by p which is to be suitably chosen later. Assume on the contrary that α_5, when its represented series is convergent, is algebraic over $F_q(x)$. Since $F_q(x)$ is an algebraic extension of $F_p(x)$, then α_5 is algebraic over $F_p(x)$, and so is a root of a linear equation of the form $\sum_{j=\ell}^{m} A_j t^{p^{\gamma j}} = 0$, $A_\ell \neq 0$, $A_m \neq 0$, $A_j \in F_p[x]$. Substituting for t by α_5 and multiplying by $G^{\beta^\gamma p^{\gamma\ell}} M^{d_\beta p^{\gamma m}}$, we get $0 = I + Q$, where

$$I := G^{\beta^\gamma p^{\gamma\ell}} M^{d_\beta p^{\gamma m}} \sum_{j=\ell}^{m} A_j \sum_{k=0}^{k_j} \left(\frac{P_k}{Q_k}\right)^{p^{\gamma j}} \frac{1}{G^{k^\gamma p^{\gamma j}}},$$

$$Q := G^{\beta^\gamma p^{\gamma\ell}} M^{d_\beta p^{\gamma m}} \sum_{j=\ell}^{m} A_j \sum_{k \geq k_j+1} \left(\frac{P_k}{Q_k}\right)^{p^{\gamma j}} \frac{1}{G^{k^\gamma p^{j\gamma}}},$$

and $k_j := \left[\frac{\beta}{p^{j-\ell}}\right]$ is a non-increasing function of j.

Step 1. We claim that I is integral.

Since the power of G appearing in I is $\geq \beta^\gamma p^{\gamma\ell} - k_j^\gamma p^{\gamma j} \geq$

$\beta^\gamma p^{\gamma\ell} - \frac{\beta^\gamma}{p^{(j-\ell)\gamma}} p^{\gamma j} \geq 0$ ($j = \ell,\ldots,m$), and

$\deg Q_k^{p^{\gamma j}} \leq d_{k_j} p^{\gamma j} \leq d_{k_\ell} p^{\gamma m} = d_\beta p^{\gamma m}$, then I is integral.

Step 2. We claim that $I \neq 0$ when β is sufficiently large and not divisible by p.

Let $\mu := \min_{j=\ell+1,\ldots,m} (\beta^\gamma p^{\gamma\ell} - k_j^\gamma p^{\gamma j}, \beta^\gamma p^{\gamma\ell} - (\beta-1)^\gamma p^{\gamma\ell})$.

Now $k_j := \left[\frac{\beta}{p^{j-\ell}}\right] = \frac{\beta - \eta_j}{p^{j-\ell}}$, where

$1 \leq \eta_j \leq p^{j-\ell} - 1$ ($j = \ell+1,\ldots,m$). Consider each term in the definition of μ. We see that

$\beta^\gamma p^{\gamma\ell} - k_j^\gamma p^{\gamma j} = \beta^\gamma p^{\gamma\ell} - (\beta - \eta_j)^\gamma p^{\gamma\ell} \geq \beta^\gamma p^{\gamma\ell} - (\beta - 1)^\gamma p^{\gamma\ell}$.

Thus $\mu = \beta^\gamma p^{\gamma\ell} - (\beta - 1)^\gamma p^{\gamma\ell} = O(\beta^{\gamma-1}) \to \infty$ ($\beta \to \infty$) because $\gamma > 1$. From the definition of I, we get

$$I = M^{d_\beta p^{\gamma m}} G^{\beta^\gamma p^{\gamma\ell}} \left\{A_\ell \sum_{k=0}^{k_\ell} \left(\frac{P_k}{Q_k}\right)^{p^{\gamma\ell}} \frac{1}{G^{k^\gamma p^{\gamma\ell}}} + \ldots + A_m \sum_{k=0}^{k_m} \left(\frac{P_k}{Q_k}\right)^{p^{\gamma m}} \frac{1}{G^{k^\gamma p^{\gamma m}}}\right\}$$

Observe that after distributing the multiplier inside ,the lowest power of G in each sum within the bracket is minimal when $k = k_\ell, \ldots, k_m$, respectively. If $k = k_\ell$, the lowest power of G is $\beta^\gamma p^{\gamma\ell} - k_\ell^\gamma p^{\gamma\ell} = 0$ in the first sum. If $k = k_{\ell+1}$, the lowest power of G is $\beta^\gamma p^{\gamma\ell} - k_{\ell+1}^\gamma p^{\gamma(\ell+1)} \geq \mu$ (by the definition of) in the second sum.

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If $k = k_m$, the lowest power of G is $\beta^\gamma p^{\gamma\ell} - k_m^\gamma p^{\gamma\ell} \geq \mu$ in the last sum. Note also that in the first sum the power of G next to the lowest power is $\beta^\gamma p^{\gamma\ell} - (k_\ell - 1)^\gamma p^{\gamma\ell} = \beta^\gamma p^{\gamma\ell} - (\beta - 1)^\gamma p^{\gamma\ell} \geq \mu$. Therefore,

$$I \equiv M^{d_\beta p^{\gamma m}} A\left(\frac{P_\beta}{Q_\beta}\right)^{p^{\gamma\ell}} \not\equiv 0 \pmod{G^\mu}, \text{ by (ii), (iii), (iv)}$$

and (v), for infinitely many β not divisible by p. This yields $I \neq 0$ when β is sufficiently larger and not divisible by p.

Step 3. We claim that deg $Q \to -\infty$ ($\to \infty$).

Let N be an arbitrary term of Q.Then $N = M^{d_\beta p^{\gamma m}}$

$$G^{\beta^\gamma p^{\gamma\ell}} A_j\left(\frac{P_k}{Q_k}\right)^{p^{\gamma j}} \frac{1}{G^{k^\gamma p^{\gamma j}}} \quad (j = \ell,\dots,m \; ; \; k \geq k_j + 1)$$

and so

$\deg N \leq m_0 d_\beta p^{\gamma m} + g\,\beta^\gamma p^{\gamma\ell} + a + p^{\gamma j} (\deg P_k - \deg Q_k) - g\,k^\gamma p^{\gamma j}$,

where $g := \deg G$, $a := \max_j \deg A_j$, $m_0 := \deg M$

$$\leq a + g\,\beta^\gamma p^{\gamma\ell} + o(\beta^{\gamma-1}) + p^{\gamma j} o(k^{\gamma-1}) - g\,k^\gamma p^{\gamma j} \leq o(\beta^{\gamma-1})$$

$$+ g\,\beta^\gamma p^{\gamma\ell} - g\,p^{\gamma j}(k_j + 1)^\gamma \left\{1 - \frac{o(k_j^{\gamma-1})}{k_j^\gamma}\right\}$$

$$= o(\beta^{\gamma-1}) + g\,\beta^\gamma p^{\gamma\ell} - g p^{\gamma\ell}(\beta - \eta_j + p^{j-\ell})^\gamma \left\{1 - \frac{o(\beta^{\gamma-1})}{\beta^\gamma}\right\} \leq$$

$$o(\beta^{\gamma-1}) + g\,\beta^\gamma p^{\gamma\ell} - g\,p^{\gamma\ell}(\beta + 1)^\gamma \left\{1 - \frac{o(\beta^{\gamma-1})}{\beta^\gamma}\right\}$$

$$\leq o(\beta^{\gamma-1}) - g\,p^{\gamma\ell}\beta^{\gamma-1}\left\{ \left(\gamma - \frac{o(\beta^{\gamma-1})}{\beta^\gamma}\right) + \text{lower terms} \right\}$$

$\to -\infty$ ($\beta \to \infty$).

From $0 = I + Q$, steps 1 and 3, we conclude that $I = Q = 0$, which contradicts step 2. Hence α_5 is transcendental over $F_q(x)$ as to be proved.

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