ScienceAsia 26 (2000) : **21-31**

Convexity Properties of Nakano spaces

Sompong Dhompongsa
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.

Received 11 Mar 1999
Accepted 31 Oct 2000

ABSTRACT Several well-known geometrical properties are investigated on the Nakano sequence spaces. It is interesting to see how these properties can be described in terms of the given sequence $\{p_k\}$. The results also supply a good source of examples supplement to the Orlicz sequence spaces.

KEYWORDS: Nakano sequence spaces, convexity.

INTRODUCTION

Let X be a Banach space, and let S (X) and B (X) denote its unit sphere and unit ball, respectively. An element e in B (X) is called an **extreme** point if there are no distinct elements x,y in B (X) with 2e = x + y. We write Ext B (X) for the set of all extreme points in B (X). If Ext B (X) = S (X), then X is called a **rotund** (R) space. For each x in B (X), write

$\lambda(x) = \sup\{\lambda \in [0,1]: x = \lambda e + (1-\lambda)y$ for some $e \in \text{Ext } B(X), y \in B(X)\}$.

If $\lambda(x) > 0$ for all $x \in B(X)$, then X is said to have the **λ - property**. Moreover, if

$\lambda(X) := \inf\{\lambda(x) : x \in S(X)\} > 0$,

then X is said to have the **uniform λ - property**.

We shall study many more properties on convexity of Banach spaces. For simplicity of the presentation, we state each definition in each of its corresponding section to follow. References for various kinds of geometric properties can be found in, for examples, Day[5] and Chen.[2] Another reference that contains a number of examples which separate various convexity properties is M. A. Smith.[15] At first, we present their connections in Figure 1. The sign "- - - -►" indicates that the implication always holds in general, whereas "——►" holds for Nakano sequence spaces. Then we summarize the results in Table 1.

For $k \geq 0$, let μ_k and ν_k stand for the following conditions.

μ_k : $\mu\{n : p_n = 1\} \leq k$.

ν_k : $\inf_{n \notin F} p_k > 1$ for some finite set F having at most k elements.

Then, for Nakano spaces,

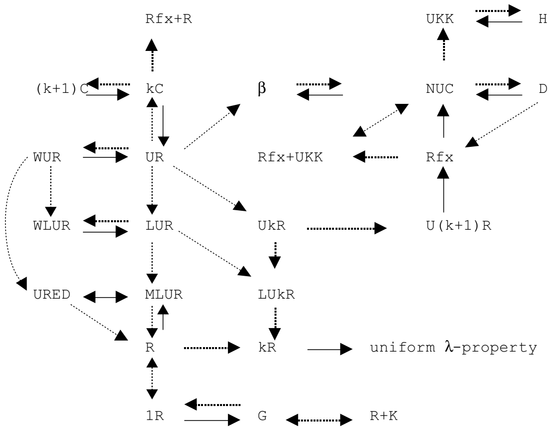

Fig 1.

Table 1.

ℓ always has H, K and UKK
LUR ⇔ WLUR ⇔ μ_0 or ν_1
LUkR ⇔ μ_0 or ν_k
R ⇔ MLUR ⇔ URED ⇔ G ⇔ μ_1
kR ⇔ μ_k
Rfx ⇔ β ⇔ NUC ⇔ D ⇔ ν_k for some k
UR ⇔ kC ⇔ WUR ⇔ ν_1
UkR ⇔ ν_k
uniform λ-property ⇔ μ_k for some k.

The above geometric properties have been thoroughly discussed in the literature, especially on the Orlicz function spaces and Orlicz sequence spaces endowed with both the Orlicz and the Luxemburg norms. In this paper, we extend the study on another sequence spaces which we describe now.

Let $\{p_k\}$ be a sequence of positive real numbers larger than or equal to one. Denote $\ell := \ell^{(p_k)}$ for the Banach space of all real sequences $x = (x_k)$ such that $\rho(\lambda x) < \infty$ for some $\lambda > 0$, where $\rho(x)$ is the modular of x defined by

$$\rho(x) = \sum_{k=1}^{\infty} |x_k|^{p_k},$$

with norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho\left(\tfrac{x}{\lambda}\right) \le 1\right\}.$$

The space is called a **Nakano** sequence space.

For $0 < p_k \le 1$, the space ℓ has been studied by various authors, eg.[10, 12, 8-9] The norm so defined, which may be called the Luxemburg norm, differs from the one introduced by Nakano. In fact, Nakano[12] defined the norm of x as

$$\|x\| = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty} \frac{1}{p_k} \left|\frac{x_k}{\lambda}\right|^{p_k} \le 1\right\}.$$

Note that, for each $x \in \ell$, $\sum_{k=1}^{\infty} \frac{1}{p_k} |\lambda x_k|^{p_k} < \infty$ for some $\lambda > 0$. Nakano sequence spaces are special cases of Musielak-Orlicz sequence spaces. Results in the paper may give some ideas of how ones should formulate the corresponding results for the more general cases. In Chen[2], extreme points as well as rotundity and uniform rotundity are studied for Musielak-Orlicz function spaces.

From now on, we assume that the sequence $\{p_k\}$ is bounded. The following observation will be needed throughout the paper:

(1) $\lim_{\delta \to 1} \rho(\delta x) = \rho(x)$

(2) $\|x\|^{p_*} \le \rho(x) \le \|x\|^{p^*}$ if $\|x\| \ge 1$,

$\|x\|^{p^*} \le \rho(x) \le \|x\|^{p_*}$ if $\|x\| < 1$,

where $p_* = \inf_k p_k$ and $p^* = \sup_k p_k$,

(3) $\|x\| = 1 \Leftrightarrow \rho(x) = 1$,

and

$\|x_n\| \to 1 \Leftrightarrow \rho(x_n) \to 1$.

We also apply, from time to time, the following notations: For each n, e_n is the standard vector in ℓ defined by $e_n = (\delta_{mn})_{m=1}^{\infty}$.

For a vector $x = (x_k)$ in ℓ, we write for each n,

$$x(n) := \sum_{k=1}^{n} x_k e_k,$$

$$x(n,\infty) := \sum_{k>n} x_k e_k.$$

Results In what follows, X will stand for a Banach space.

Extreme points and rotundity

Lemma 1 $x \in \operatorname{Ext} B(\ell)$ if and only if
(i) $\rho(x) = 1$, and
(ii) $\mu\{k : x_k \ne 0 \text{ and } p_k = 1\} \le 1$,
where μ is the counting measure on Z^+

Proof ($\Rightarrow$) It is clear that each e_n satisfies (i) and (ii). Now let x be an extreme point with $\mu\{k : x_k \ne 0\} \ge 2$. Suppose, without loss of generality, that $p_1 = p_2 = 1$, $x_1 \ne 0$ and $x_2 \ne 0$. Thus $|x_1|, |x_2| \in (0,1)$. Choose $\varepsilon > 0$ such that $|x_1| - \varepsilon > 0$ and $|x_2| - \varepsilon > 0$. Let $y = (y_k)$ and $z = (z_k)$ where

$$(Y_k, Z_k) = \begin{cases} (x_k + \varepsilon \operatorname{sgn} x_k,\ x_k - \varepsilon \operatorname{sgn} x_k), & \text{if } k = 1, \\ (x_k - \varepsilon \operatorname{sgn} x_k,\ x_k + \varepsilon \operatorname{sgn} x_k), & \text{if } k = 2, \\ (x_k, x_k), & \text{otherwise.} \end{cases}$$

Hence $2x = y + z$, $\|y\| \le 1$, $\|z\| \le 1$, a contradiction and we have (ii). To prove (i) we suppose on the contrary that

$$\lim_{\delta \uparrow 1} \rho(\delta x) = r < 1.$$

Choose $\varepsilon > 0$ so small that $|x_1 \pm \varepsilon|^{p_1} < |x_1|^{p_1} + \frac{1-r}{2}$.

Let $y_1 = x_1 + \varepsilon$, $z_1 = x_1 - \varepsilon$, and $y_k = x_k = z_k$ for all $k \ge 2$. Then $2x = y + z$,

$$\rho(\delta y),\ \rho(\delta z) < \rho(\delta x) + \frac{1-r}{2}$$

for all $\delta < 1$. By (1) we have $\|y\| \le 1$ and $\|z\| \le 1$, a contradiction.

($\Leftarrow$) Assume that x satisfies (i) and (ii). Suppose that $2x = y + z$ for some $y, z \in B(\ell)$. Thus $x_k \ne 0$ for at least 2 k, say $x_1 \ne 0$, $x_2 \ne 0$, and then $p_1 \ne 1$ or $p_2 \ne 1$, say $p_1 \ne 1$. For some ε, $y_1 = x_1 + \varepsilon$, $z_1 = x_1 - \varepsilon$, and say,

$$|x_1|^{p_1} < \frac{|y_1|^{p_1} + |z_1|^{p_1}}{2}.$$

Write $p > 0$ for the difference of these two numbers. Now

$$\frac{\rho}{2} + \rho(\delta x) < \sum_{k=1}^{\infty} \frac{|\delta y_k|^{p_k} + |\delta z_k|^{p_k}}{2} \le \frac{1+1}{2} = 1$$

for all $0 < \delta < 1$. This implies $\rho(\delta x) < 1 - \frac{\rho}{2}$ for all $0 < \delta < 1$, and therefore,

$$\lim_{\delta\uparrow 1}\rho(\delta x) < 1$$

contradicting (i).

We now immediately have

Theorem 2 ℓ is rotund if and only if $p_k = 1$ for at most one k.

We consider now a more general type of rotundity.

For $k \geq 1$, X is a **k-rotund** (kR) space if for vectors $x_1, \ldots, x_{k+1}$ in S (X) with $\|x_1 + \ldots + x_{k+1}\| = k + 1$, the set $\{x_1, \ldots, x_{k+1}\}$ is linearly dependent.

Theorem 3 ℓ is kR if and only if $\mu\{k : p_k = 1\} \leq k$.

Proof ($\Rightarrow$) Suppose $p_1 = \ldots = p_{k+1} = 1$. Consider the independent set $\{e_1, \ldots, e_{k+1}\}$ of elements in S (ℓ). We see that $\|e_1 + \ldots + e_{k+1}\| = k + 1$.

($\Leftarrow$) Suppose $\|x_1\| = \ldots = \|x_{k+1}\| = 1$, $\|x_1 + \ldots + x_{k+1}\| = k + 1$ and $x_1, \ldots, x_{k+1}$ are linearly independent. Writing $x_i = (x_{i1}, x_{i2}, \ldots)$, by independence of x_i, there are $j_1, \ldots, j_{k+1}$ such that for each j_m, $x_{ij_m} \neq 0$ for some i. Otherwise,

$$\det\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+1}\end{bmatrix} = 0$$

a contradiction.

Suppose $j_m = m$ for $m = 1, \ldots, k + 1$. Note that $p_m \neq 1$ for some m, say, $p_1 \neq 1$. Put

$$x_0 = x_1 + \ldots + x_{k+1}.$$

Then $\|x_0\| = k + 1$, $\|\frac{x_0}{k+1}\| = 1$, and $\rho(\frac{x_0}{k+1})$ 1. Now

$$1 = \sum_j \left|\frac{x_{0j}}{k+1}\right|^{p_j} = \left|\frac{x_{01}}{k+1}\right|^{p_1} + \sum_{j>1}\left|\frac{x_{0j}}{k+1}\right|^{p_j}$$

$$< \frac{|x_{11}|^{p_1} + |x_{21}|^{p_1} + \ldots + |x_{k+1,1}|^{p_1}}{k+1} + \sum_{j>1}\frac{|x_{1j}|^{p_j} + \ldots + |x_{k+1,j}|^{p_j}}{k+1}$$

$$= \sum_{j\geq 1}\frac{|x_{1j}|^{p_j} + \ldots + |x_{k+1,j}|^{p_j}}{k+1} = \frac{\rho(x_1) + \ldots + \rho(x_{k+1})}{k+1} = 1,$$

a contradiction.

Remark 4 R $\Leftrightarrow$ 1R.

Proof ($\Rightarrow$) If $\|x_1\| = \|x_2\| = 1$ and $\|x_1 + x_2\| = 2$, then $x := (x_1 + x_2)/2 \in$ S (ℓ). Now $2x = x_1 + x_2$ implies $x_1 = x_2$, and thus x_1 and x_2 are linearly dependent.

($\Leftarrow$) Let $\|x\| = 1$, $\|y\| \leq 1$, $\|z\| \leq 1$, and $2x = y + z$. From $2 = \|y + z\| \leq \|y\| + \|z\| \leq 2$, we have $\|y\| = \|z\| = 1$, and thus $y = cz$ for some c. Now $1 = \|y\| = |c|\|z\| = |c|$, we have $c = 1$.

From Remark 4 we see that Theorem 2 becomes a corollary of Theorem 3.

UNIFORM λ-PROPERTY

Recall that

(4) $\lambda(\ell) = \inf\{\ \ell(x) : \rho(x) = 1\}$.

Theorem 5 ℓ has the uniform λ-property if and only if $\mu\{k : p_k = 1\} < \infty$. In general, we have $\lambda(\ell) = 1/\mu\{k : pk = 1\}$.

Proof Put $w = \mu\{k : p_k = 1\}$.

($\Rightarrow$) Suppose $w = \infty$, i.e. $p_k = 1$ for infinitely many k. For convenience assume $p_k = 1$ for all k. Let $0 < r < 1$ and choose $x = (x_k)$ such that $0 < x_k \leq r$ for all k and $\rho(x) = 1$. If $x = \lambda a + (1-\lambda) y$ for some $a = (a_k) \in$ Ext B (ℓ), $y \in$ B (ℓ) and some $\lambda \in (0,1]$, then, by Lemma 1, $a_{k_0} = 1$ for some k_0 and $a_k = 0$ otherwise. We see that $\lambda \neq 1$ since x is not an extreme point. Now

$$\sum_{k\neq k_0}|y_k| = \frac{1}{1-\lambda}\sum_{k\neq k_0}x_k = \frac{1-x_{k_0}}{1-\lambda}.$$

Since $y \in$ B (ℓ), $\frac{1-x_{k_0}}{1-\lambda} + |y_{k_0}| \leq 1$. From this we have $\lambda \leq x_{k_0} \leq r$. Therefore $\lambda(x) \leq r$, and then $\lambda(\ell) = 0$, a contradiction.

($\Leftarrow$) In case $w = 1$ the assertion is clear since S (ℓ) = Ext B (ℓ). Suppose $p_1 = \ldots = p_w = 1$ and $p_k > 1$ ($k > w > 1$). To show first $\lambda(\ell) \geq \frac{1}{w}$. Take any x with $\rho(x) = 1$. Suppose, for convenience, that $|x_1| = \max_{1\leq k\leq w}|x_k|$. Put $a = |x_1| + \ldots + |x_w|$. If $a = 0$, then $x \in$ Ext B (ℓ) and $\lambda(x) = 1$. Otherwise, put $\lambda = \frac{|x_1|}{a}$. If $\lambda = 1$, again $x \in$ Ext B (ℓ) and we are done. Now put

$$y = (0, \frac{x_2}{1-\lambda}, \ldots, \frac{x_w}{1-\lambda}, x_{w+1}, x_{w+2}, \ldots)$$

and

$$c = (b, 0, \ldots, 0, x_{w+1}, x_{w+2}, \ldots),$$

where $c_k = 0$ for $2 \le k \le w$, $|b| = a$, and $\frac{|x_1|}{a} b = x_1$.

Thus

$$x = \lambda c + (1-\lambda) y,$$

$$\rho(c) = |b| + \sum_{k>w} |x_k|^{p_k} = \sum_{k\ge 1} |x_k|^{p_k} = 1, \quad c \in \text{Ext } B(\ell),$$

and

$$\rho(y) = \frac{|x_2| + \ldots + |x_w|}{1-\lambda} + \sum_{k>w} |x_k|^{p_k} = \frac{a - |x_1|}{1-\lambda} + \sum_{k>w} |x_k|^{p_k}$$

$$= a + \sum_{k>w} |x_k|^{p_k} = \rho(x) = 1.$$

Thus $\lambda(x) \ge \lambda = \frac{|x_1|}{a} \ge \frac{1}{w}$, and by (4) $\lambda(\ell) \ge \frac{1}{w}$.

To complete the proof, we now show that $\lambda(\ell) \le \frac{1}{w}$. Let

$x = \sum_{k=1}^{w} \frac{1}{w} e_k$. If $x = \lambda c + (1-\lambda) y$ for some $\lambda \in [0,1)$, $c \in \text{Ext } B(\ell)$ and $y \in B(\ell)$, then $c_1 \ne 0$, say, and $c_k = 0$ otherwise. Also for some a_k,

$$y = \sum_{k+1}^{w} a_k e_k, \frac{1}{w} \ge a_1 = \frac{\frac{1}{w} - \lambda}{1-\lambda}, a_2 = \ldots = a_w = \frac{1}{w(1-\lambda)}.$$

Since $\|y\| \le 1$,

$$\rho(y) = \frac{\frac{1}{w} - \lambda}{1-\lambda} + \frac{w-1}{w(1-\lambda)} \le 1.$$

Note that $a_1 \ge 0$. Now

$$\lambda = \frac{\frac{1}{w} - a_1}{1-a_1} \le \frac{1}{w}.$$

Thus $\lambda(x) \le \frac{1}{w}$, and we have $\lambda(\ell) \le \frac{1}{w}$.

H-PROPERTY

X is said to have the **property (H)** if each point of S (X) is an H-point of B (X), that is, every weak convergence of points x_n in B (X) to a point in S (X) with $\|x_n\| \to 1$ is a convergence in norm.

Theorem 6 ℓ has property H.

Proof Let $x_0 \in S(\ell)$, $x_n \in B(\ell)$ be such that $x_n \xrightarrow{w} x_0$. We observe that

(a) $x_{nk} \to x_{0k}$ for all k,

and

(b) $\|x_n\| \to \|x_0\| = 1$.

To show $x_n \to x_0$, we show that for each $\lambda \in (0,1)$, there exists N_λ such that

$$\text{(c)} \quad \sum_{k>N_\lambda} \left| \frac{x_{nk} - x_{0k}}{\lambda} \right|^{p_k} \le 1$$

for all large n. Obviously, since $x_n \to x_0$ pointwise, the convergence $x_n \to x_0$ follows from (c). To prove (c) we suppose on the contrary that there exists an increasing sequence of natural numbers N_n, $N_n \to \infty$, and $\lambda_0 \in (0,1)$ such that

$$\sum_{k>n} \left| \frac{x_{N_n k} - x_{0k}}{\lambda_0} \right|^{p_k} > 1$$

for all n. Thus $\sum_{k>n} |x_{N_n k} - x_{0k}|^{p_k} > \lambda_0^{p^*}$ for all n

where $p^* = \sup_k p_k$. Put $\varepsilon_0 = \lambda_0^{p^*} / 4^{p^*+1}$, $y_n = (x_n + x_0)/2$. Choose N_0 so that

$$\sum_{k>N_0} |x_{0k}|^{p_k} < \varepsilon_0.$$

Thus

$$\sum_{k\le N_0} |x_{0k}|^{p_k} \ge 1 - \varepsilon_0,$$

and thus for all $n \ge N_0$ we have

$$\text{(d)} \quad \sum_{k>n} |x_{N_n k}|^{p_k} \ge \sum_{k>n} \frac{|x_{N_n k} - x_{0k}|^{p_k}}{2^{p^*}} - \sum_{k>n} |x_{0k}|^{p_k} \ge 4\varepsilon_0 - \varepsilon_0 = 3\varepsilon_0.$$

Choose n_0, by (a), so large that for each $n \ge n_0$,

$$\text{(e)} \quad \sum_{k\le N_0} |x_{nk}|^{p_k} > \sum_{k\le N_0} |x_{0k}|^{p_k} - \varepsilon_0 > 1 - 2\varepsilon_0.$$

Take $n' > N_0$ so that $N_{n'} > n_0$ and (d) holds for n'. Therefore

$$\text{(f)} \quad \rho(x_{N_{n'}}) = \left(\sum_{k\le N_0} + \sum_{k>N_0}\right) \left| x_{N_{n'} k} \right|^{p_k}$$

$$\ge \left(\sum_{k\le N_0} + \sum_{k>n'}\right) \left| x_{N_{n'} k} \right|^{p_k}$$

$$> (1 - 2\varepsilon_0) + 3\varepsilon_0 = 1 + \varepsilon_0,$$

a contradiction. Hence we have (c).

Remark 7 The proof of Theorem 6 yields more than its statement. More precisely, it shows that if x_n (in the unit ball) converges pointwise to x (in the unit sphere), then $\|x_n - x\| \to 0$. This fact will be needed later, in the proof of Theorem 16 and 21.

Uniform convexity

X is **uniformly convex** (UC) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in S(X)$, inequality $\|x - y\| \ge \varepsilon$ implies $\frac{\|x+y\|}{2} < 1-\delta$.

Theorem 8 ℓ is UC if and only if $\inf_{k\ne k_0} p_k > 1$ for some k_0.

Lemma 9 Let $K > 1$ and $f(p) = f_K(p) = (1 + \frac{1}{K})^p + (1-\frac{1}{K})^p - 2$ $(p \ge 1)$. Then f is a strictly increasing function on $[1, \infty)$.

Proof Note that $f(1) = 0$,

$$f'(p) = (1+\frac{1}{K})^p\log((1+\frac{1}{K})+(1-\frac{1}{K})\log(1-\frac{1}{K},$$

$$f''(p) = (1+\frac{1}{K})^p \log^2 (1+\frac{1}{K}) + (1-\frac{1}{K})^p \log^2 (1+\frac{1}{K}) > 0.$$

To show $f'(1) > 0$. For then $f > 0$ on $(1, \infty)$ by convexity. Putting $y = x\log x$, we see that

$$y' = 1 + \log x, \quad y'' = \frac{1}{x} > 0.$$

The convexity implies $(1 + x)\log (1 + x)+ (1 - x)\log (1 - x) > 2 (1 \log 1) = 0$ $(0 < x < 1)$.

Remark 10 From Lemma 9 we derive the following estimations.

(a) For $0 \le \frac{a}{K} < \varepsilon \le a$, and $p \ge p_* > 1$,

$$(a + \varepsilon)^p + (a - \varepsilon)^p - 2a^p = a^p [(1 + \frac{\varepsilon}{a})^p + (1 - \frac{\varepsilon}{a})^p - 2]$$

$$\ge a^p [(1 + \frac{1}{K})^p + (1 - \frac{1}{K})^p - 2]$$

$$\ge a^p f(p_*) \ge \varepsilon^p f(p_*).$$

(b) For $0 \le a < \varepsilon \le Ka$, and $p \ge p_* > 1$,

$$(\varepsilon + a)^p + (\varepsilon - a)^p - 2a^p \ge \varepsilon^p [(1+ \frac{\varepsilon}{a})^p + (1 - \frac{\varepsilon}{a})^p - 2] \ge \varepsilon^p f(p_*).$$

(c) For $0 \le Ka < \varepsilon$, and $p > 1$,

$$(\varepsilon+ a)^p + (\varepsilon - a)^p - 2a^p \ge (\varepsilon+ a)^p + (\varepsilon - a)^p - 2 (\frac{\varepsilon}{K})^p$$

$$= \varepsilon^p [(1 + \frac{a}{\varepsilon})^p + (1 - \frac{a}{\varepsilon})^p - \frac{2}{K^p}]$$

$$\ge \varepsilon^p (2 - \frac{2}{K^p}).$$

Proof of Theorem 8

(Case $p_{n_k} \to 1$) Put $x_k = e_{n_k}, Y_k = p_{n_{k+1}}$. Thus $\|x_k - y_k\| \to 2$, but

$$\rho(\frac{x_k + y_k}{2}) = \left(\frac{1}{2}\right)^{p_{n_k}} + \left(\frac{1}{2}\right)^{p_{n_k+1}} \to 1.$$

(Case $p_1 = p_2 = 1$) This is clear, since UC implies R.

(Case $\inf_k p_k = p_* > 1$) Let $K > 1$ be a fixed number to be chosen appropriately later. Let $f = f_K$ be as defined in Lemma 9. Suppose $x_n, y_n \in S(\ell)$, $\|x_n - y_n\| \ge \varepsilon_0$ for some $\varepsilon_0 > 0$ and

$$\left\| \frac{x_k + y_k}{2} \right\| := \| z_n \| \to 1.$$

Write $z_n = (a_{nk})$, $\varepsilon_{nk} = |x_{nk} - a_{nk}|$. Thus $x_{nk} = a_{nk} \pm \varepsilon_{nk}$ and $x_{nk} = a_{nk} \pm \varepsilon_{nk}$ if and only if $y_{nk} = a_{nk} \mp \varepsilon_{nk}$. Given $\varepsilon > 0$, choose $K > 1$ so that $\frac{1}{K} < \varepsilon$.

Put $\gamma = \min \{f(p_*), 2 - \frac{2}{K}\}$. Write, for each n, $\sum_< = \sum_{<,K}$ (and $\sum_\ge$) for the summations corresponding to k for which

$$\varepsilon_{nk} < \frac{|a_{nk}|}{K} \quad (\text{respectively, } \varepsilon_{nk} \ge \frac{|a_{nk}|}{K}).$$

Note that

(d) $|a_{nk} + \varepsilon_{nk}|^{p_k} + |a_{nk} - \varepsilon_{nk}|^{p_k} - 2|a_{nk}|^{p_k} = (|a_{nk}| + \varepsilon_{nk})^{p_k} + ||a_{nk}| - \varepsilon_{nk}|^{p_k} - 2|a_{nk}|^{p_k}$,

(e) for some $\lambda_0 > 0$, $\rho(x_n-y_n) = \sum_k (2\varepsilon_{nk})^{p_k} \ge \lambda_0$ for all n,

and

(f)

$$\sum_k (|a_{nk} + \varepsilon_{nk}|^{p_k} + |a_{nk} - \varepsilon_{nk}|^{p_k} - 2|a_{nk}|^{p_k}) = \rho(x_n) + \rho(y_n) - 2\rho(Z_n) \to 0.$$

By (d), (a), (b), (c), and (f) we have

$$\sum_k (\varepsilon_{nk})^{p_k} = (\sum_< + \sum_\ge)(\varepsilon_{nk})^{p_k} \le \frac{1}{K^{p_*}} + \frac{1}{\gamma}(\rho(x_n) + \rho(y_n) - 2\rho(Z_n)) < \varepsilon + \varepsilon$$

for all large n, contradicting (e).

(Case $p_1 = 1, \ \inf_{k\geq 2} p_k = P_* > 1$) Let x_n, y_n, z_n, a_{nk}, and ε_{nk} be as above. By passing through subsequences we may assume that $x_{n1} \to a$ and $y_{n1} \to b$ for some a, b. Put $c = \frac{(a+b)}{2}$. Thus,

$$\rho\,(x_n\ (1,\ \infty)) \to A := 1 - |a|,$$
$$\rho\,(y_n\ (1,\ \infty)) \to B := 1 - |b|,$$
$$\rho\,(z_n\ (1,\ \infty)) \to C := 1 - |c|.$$

Note that A + B = 2C. As before we have,

$$\sum_{k\geq 2}(\varepsilon_{nk})^{p_k} \leq \frac{1}{K^{p_*}} + \frac{1}{\gamma}(\rho(x_n\ (1,\infty)) + \rho\ (y_n\ (1,\infty)) - 2\ \rho\ (z_n\ (1,\infty))\ )$$

which leads to

$$\text{(g)} \qquad \sum_{k\geq 2}(\varepsilon_{nk})^{p_k} \to \ \text{as}\ \ n \to \infty.$$

Since 2C = A + B, we see that |a + b| = |a| + |b|, i.e. ab > 0. Suppose $a \geq b > 0$. From (g) and (e) we obtain |a - b| > 0. Therefore $a > b \geq 0$, and so A < C < B. Choose L > 1 so that $\eta := (1+\frac{1}{L})^{p^*} - 1 < \frac{B-C}{C}$, where $p^* = \sup_k p_k$. By (g) we have

$$\text{(h)} \qquad \sum_{k\geq 2,\ \geq, L} (\,|a_{nk} + \varepsilon_{nk}|\,)^{p_k} \leq (1+L)^{p^*} \sum_{k\geq 2}(\varepsilon_{nk})^{p_k} \to 0$$
as $n \to \infty$.

On the other hand we have

$$\text{(i)} \qquad \sum_{k\geq 2,\ \geq, L} (\,|a_{nk} + \varepsilon_{nk}|\,)^{p_k} \leq (1+\eta) \sum_{k\geq 2}|a_{nk}|^{p_k} \to$$
$(1+\eta)C$ as $n \to \infty$.

But then (h) and (i) imply

$$\limsup_{n\to\infty}(\sum_{k\geq 2}(|a_{nk}| + \varepsilon_{nk})^{p_k}) \leq (1+\eta)C$$

which leads to a contradiction since

$$\limsup_{n\to\infty}(\sum_{k\geq 2}(|a_{nk}| + \varepsilon_{nk})^{p_k}) \geq \limsup_{n\to\infty}(\sum_{k\geq 2}(|y_{nk}|^{p_k} = B > (1+\eta)C.$$

Property β

X has the **property** β if for any ε > 0, there exists δ > 0 such that

$$\alpha\ (D\ (x,B\ (X))\backslash B\ (X)) < \varepsilon$$

whenever $1 < \|x\| < 1 + \delta$. Here α is the Kuratowski measure of non-compactness on subsets of X, and

$$D\ (x,B\ (X)) = \text{co}\ (\{x\} \cup B\ (X)),$$

the **drop** determined by x.

We will make use of the following equivalent form of property β.(See[9])

X has property β if and only if for each ε > 0, there exists δ > 0 such that for each x in B (X) and each sequence $\{x_n\}$ in B (X) with sep $(x_n) \geq \varepsilon$, there exists k such that $\left\|\frac{x+x_k}{2}\right\| \leq 1-\delta$.

Here

$$\text{sep}\ (x_n) := \inf_{m\neq n}\|x_m - x_n\|.$$

Theorem 11 ℓ has property β if and only if $\liminf_{k\to\infty} p_k > 1$.

Proof If $p_k \to 1$, then for each n_0, choose k_0 such that $(\frac{1}{2})^{p_k} > \frac{1}{2} - \frac{1}{n_0}$ for all $k \geq k_0$. Let $x_k = e_k$ $(k \geq k_0)$ and let $x = e_{k_0}$. We see that sep $(x_k) \geq \frac{1}{2}$. But $\left\|\frac{x+x_k}{2}\right\| \geq (\frac{1}{2})^{p_k} + (\frac{1}{2})^{p_{k_0}} > 1 - \frac{2}{n_0}$ for all $k \geq k_0$, violating the property β.

To prove the converse, suppose

$$p_1 = \ldots = P_{m_0} = 1 \ \text{ and } \inf_{k\neq 1,\ldots,m_0} P_k > 1.$$

Note that the space

$$\ell' := \ell^{(1,P_{m_0+1},P_{m_0+2},\ldots)}$$

is UC and hence has property β. Thus given ε > 0, we take a δ > 0 such that for each sequence x_n in B (ℓ') and $x \in S$ (ℓ') with sep $(x_n) \geq \frac{\varepsilon}{2}$, we have $\left\|\frac{x+x_k}{2}\right\| \leq 1-\delta$ for some k. Now let $x_n \in B$ (ℓ) with sep $(x_n) \geq \varepsilon$. Passing through a subsequence if necessary, we assume $x_{ni} \to a_i$ for each i = 1, ..., m_0. Again, passing through a tail of the sequence we assume that sep $(x'_n) \geq \frac{\varepsilon}{2}$ where $x'_n = \sum_{k\geq 1} x_{n(m_0+k)} e_{k+1}$.

Let $x_0 \in S(\ell)$ and put $a = |a_1| + ... + |a_{m_0}|$ and $b = |x_{01}| + ... + |x_{0m_0}|$. Writing, for example, (b,x'_0) for the vector $be_1 + \sum_{k\geq 1} x_{0(m_0+k)}e_{k+1}$. It is clear that $(b, x'_0) \in B(\ell)$ and sep $((a, x'_n)) \geq \frac{\varepsilon}{2}$. To see that $(a, x'_n) \in B(\ell)$ for all large n we let $\lambda > 1$ and estimate $\rho(\frac{a,x'_n}{\lambda})$. Since $x_{ni} \to a_i$ $(i = 1, ...,m_0)$, and writing $\rho(a, x'_n) = \rho(x_n) + \varepsilon_n$, we see that $\varepsilon_n \to 0$. Thus, since $\rho(\frac{x_n}{\lambda}) < 1$,

$$\rho(\frac{a,x'_n}{\lambda}) = \rho(\frac{x_n}{\lambda}) + \varepsilon_n \leq 1 \text{ for all large n as desired.}$$

Now by the definition of δ we have for infinitely many k,

$$\left\| \frac{(b,x'_0)+(a,x'_k)}{2} \right\| \leq 1-\delta.$$

If $\lambda > 1 - \delta$, then $\left| \frac{a+b}{2\lambda} \right| + \rho\left(\frac{x'_0 + x'_k}{2\lambda} \right) < 1$, which in turn implies

$$\frac{\left|\frac{a_1+x_{01}}{2}\right| + ... + \left|\frac{a_{m_0}+x_{0m_0}}{2}\right|}{\lambda} + \rho\left(\frac{x'_0 + x'_k}{2\lambda} \right) < 1.$$

This means that for sufficiently large k,

$$\left\| \frac{x_0 + x_k}{2} \right\| \leq 1-\delta.$$

Uniform Kadec-Klee property

X is said to have the **uniform Kadec-Klee property** (UKK) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_n \in B(X)$ with $x_n \to x$ weakly and sep $(x_n) \geq \varepsilon$, we have $\|x\| < 1 - \delta$.

Theorem 12 ℓ has property UKK.

Proof The proof of this Theorem follows the same lines as of the proof of [1, Theorem 3.17]. We repeat here just to present some (minor) differences. Let $0 < \varepsilon < 1$ and put

$$\beta = \left(\frac{\varepsilon}{4}\right)^{p^*}$$

where $p^* = \sup_k p_k$. Choose $0 < \delta < 1$ so that $(1 - \delta)^{p^*} > 1 - \beta$. Now if $x_n \in B(\ell)$ with sep $(x_n) \geq \varepsilon$ and $x_n \xrightarrow{w} x$, we show that $\|x\| < 1 - \delta$. Suppose not, we choose $K \in \mathbb{Z}^+$ such that $\|x(K)\| > 1 - \delta$. Recall that $x(K)$ denotes the truncation of x at K, that is, $x(K) = \sum_{k=1}^{K} x_k e_k$, whereas its complement $x(K, \infty)$ is the vector $\sum_{k>K} x_k e_k$. Next we choose N such that

$\|x_n(K)\| > 1 - \delta$ and $\|(x_m - x_n)(K)\| \leq \frac{\varepsilon}{2}$ $(m, n > N, m \neq n)$. This can be done since $x_n \to x$ pointwise. The first inequality implies, by (3), that $\rho(x_n(K)) > 1 - \beta$, while the second one implies $\rho((x_n - x_m)(K, \infty)) \geq \frac{\varepsilon}{2}$. From the last estimation we may assume $\|x_n(K, \infty)\| \geq \frac{\varepsilon}{4}$. Again, by (3), we have

$$\rho(x_n(K, \infty)) \geq \|x_n(K, \infty)\|^{p^*} \geq (\frac{\varepsilon}{4})^{p^*} = \beta.$$

Thus, $\rho(x_n) > 1$, a contradiction.

Nearly uniform convexity

X is **nearly uniformly convex** (NUC) if for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that for every sequence $\{x_n\}$ in $B(X)$ with sep $(\{x_n\}) \geq \varepsilon$, we have

$$\text{co}(\{x_n\}) \cap (1 - \delta)B(X) \neq \varnothing.$$

Theorem 13 [3,Theorem] ℓ is NUC if and only if $\liminf_{k\to\infty} p_k > 1$.

Huff[8] proved that X is NUC if and only if X is reflexive and X has the property UKK. Thus, by Theorem 12 and 13, we have the following corollary.

Corollary 14 ℓ is reflexive if and only if $\liminf_{k\to\infty} p_k > 1$.

Uniform k - rotundity

X is a **uniformly rotund** (UR) space if for any $x_n, y_n \in B(X)$, $\|x_n + y_n\| \to 2$ implies $x_n - y_n \to 0$.

X is **uniformly k-rotund** (UkR) ($k \ge 1$) provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that any $x_0, x_1, \dots, x_k \in B(X)$, the inequality

$$\| x_0 + x_1 + \dots + x_k \| \ge (k+1) - \delta$$

implies $\Delta(x_0, x_1, \dots, x_k) < \varepsilon$, where

$$\Delta(x_0, x_1, \dots, x_k) = \sup_{f_i \in B(X^*)} \Delta(x_0, x_1, \dots, x_k; f_1, f_2, \dots, f_k),$$

and

$$\Delta(x_0, x_1, \dots, x_k; f_1, f_2, \dots, f_k) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_0) & f_1(x_1) & \cdots & f_1(x_k) \\ \vdots & \vdots & \cdots & \vdots \\ f_k(x_0) & f_k(x_1) & \cdots & f_k(x_k) \end{bmatrix}.$$

X is **LukR** ($k \ge 1$) if for any $x \in S(X)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $x_1, \dots, x_k \in B(X)$ with

$$\| x + x_1 + \dots + x_k \| \ge (k+1) - \delta,$$

we have $\Delta(x, x_1, \dots, x_k) < \varepsilon$.

Local uniform rotundity

X is a **locally uniform rotund** (LUR) space if every point of S (X) is a URP of B (X), that is, for each $x \in S(X)$, if $x_n \in B(X)$ and $\|x_n + x\| \to 2$, then $x_n \to x$.

Replacing the convergence in norm by the weak one for each $x \in S(X)$, we obtain a **weakly locally uniformly rotund** (WLUR) space.

Clearly, UC and UR are same property. Also LU1R $\Leftrightarrow$ LUR.

Theorem 15 ℓ is UkR if and only if $\inf_{n \notin F} p_n > 1$ $\inf_{n \notin F} P_n$ for some finite set F having at most k elements.

Proof (Case $p_1 = p_2 = \dots = p_k = p_{k+1} = 1$) Let $x_0 = e_1$, $x_1 = e_2, \dots, x_k = e_{k+1}$. It is seen that $\| x_0 + x_1 + \dots + x_k \| = k + 1$. Write π_i for the i^{th} projection on ℓ. Therefore, $\Delta(x_0, x_1, \dots, x_k; \pi_1, \pi_2, \dots, \pi_k) = 1$.

(Case $p_{n'} \to 1$ for some subsequence $\{p_{n'}\}$) Assume, for convenience, that $p_n \to 1$. Consider the sequence $\{e_n\}$. We have $\| e_n + e_{n+1} + \dots + e_{n+k} \| \to k+1$ as $n \to \infty$, whereas $\Delta(e_n, e_{n+1}, \dots, e_{n+k}; \pi_{n+1}, \pi_{n+2}, \dots, \pi_{n+k}) = 1$.

The above two examples show that ℓ is not UkR in the first two cases.

(Case $\inf_{n \ne 1, \dots, k_0} p_n > 1$ for some $k_0 \le k$) Let $x^n_0, x^n_1, \dots, x^n_k \in B(\ell)$ and

$$\| x^n_0 + x^n_1 + \dots + x^n_k \| \to k+1 \text{ as } n \to \infty.$$

To show $\Delta(x^n_0, x^n_1, \dots, x^n_k) \to 0$, we shall prove that

(a) $\displaystyle\sum_{j=1}^{k} (x^n_{ij} - x^n_{0j}) \to 0$ $(i = 1, \dots, k)$,

and

(b) $\rho((x^n_i - x^n_0)(k_0, \infty)) \to 0$ $(i = 1, \dots, k)$.

Observe that, for $f_1, \dots, f_k \in B(\ell^*)$,

(c)

$$\Delta(x^n_0, x^n_1, \dots, x^n_k; f_1, f_2, \dots, f_k) \le \det \begin{bmatrix} f_1(x^n_{11} - x^n_{01}) & \cdots & f_1(x^n_{1k} - x^n_{0k}) \\ \vdots & \cdots & \vdots \\ f_k(x^n_{11} - x^n_{01}) & \cdots & f_k(x^n_{1k} - x^n_{0k}) \end{bmatrix}$$

$$+ M \sum_{j=1}^{k} \rho((x^n_j - x^n_0)(k_0, \infty)),\ k_0, \infty)),$$

where $M = k!2^k$.

Taking (a) and (b) for granted we see that

$$\Delta(x^n_0, x^n_1, \dots, x^n_k) \to 0 \text{ as } n \to \infty.$$

Note from $\| x^n_0 + x^n_1 + \dots + x^n_k \| \to k+1$, that

$$\| x^n_i \| \to 1,\ \| x^n_i + x^n_j \| \to 2 \text{ for all } i, j.$$

Now (b) is easily obtained as in the proof of Theorem 8. Again, as in the proof of (h) and (i) in the proof of Theorem 8, we obtain from (b) that

$$\rho(x^n_i(k_0, \infty)) - \rho(x^n_0(k_0, \infty)) \to 0 \text{ as } n \to \infty$$
$(i = 1, \dots, k)$.

Now (a) is immediate.

Theorem 16 ℓ is LUR if and only if $p_k > 1$ for all k or $\inf_{k \ne k_0} p_k > 1$ some k_0.

Lemma 17 If $x_0 \in S(\ell)$, $x_n \in B(\ell)$ for all n, and $\| x_n + x_0 \| \to 2$, then $x_{nk} \to x_{0k}$ for each k where $p_k > 1$.

Proof Put $y_n = (x_n + x_0)/2$. Suppose $p_1 > 1$. If, for some $\varepsilon_0 > 0$, $|x_{n1} - x_{01}| \ge \varepsilon_0$ for infinitely many n, then there exists $\lambda_0 > 0$ such that

$$|y_{n1}|^{p_1} + \lambda_0 \le \frac{|x_{n1}|^{p_1} + |x_{01}|^{p_1}}{2} \text{ for all such } n.$$

Thus

$$\rho(y_n) + \lambda_0 \le (\rho\, x_n) + \rho(x_0))/2 \le 1,$$

which implies

$$\|y_n\| \not\to 1,$$

a contradiction.

Lemma 18 For $p_1 = 1$ and $\inf_{k\ge 2} p_k > 1$, if the norms of y_n in the proof of Lemma 17 converge to 1, then $x_{n1} \to x_{01}$.

Proof The proof is the same as the proof of Theorem 8 (Case $p_1 = 1$, $\inf_{k\ge 2} p_k > 1$). Observe that $\|x_n\| \to 1$.

Proof of Theorem 16

Suppose there are 2 p_k, say p_1 and p_2 that are equal to 1. Then ℓ is not R by Theorem 2.

Next suppose $p_1 = 1$, say, and $p_k \to 1$, otherwise pass to a subsequence. Let $x_0 = e_1$, $x_n = e_n$. Thus x_0, $x_n \in S(\ell)$,

$$Y_n := \frac{x_n + x_0}{2} = \frac{1}{2}e_1 + \frac{1}{2}e_n, \rho(Y_n) = \frac{1}{2} + (\frac{1}{2})P_n \to 1.$$

Therefore $\|x_n + x_0\| = 2\|y_n\| \to 2$, but $x_n \not\to x_0$.

Conversely, suppose $p_k > 1$ for all k or $\inf_{k\ne k_0} p_k > 1$ for some k_0. Let $x_0 \in S(\ell)$, $x_n \in B(\ell)$ and $\|x_n + x_0\| \to 2$. By Lemma 17 and 18 we have

$$X_{nk} \to x_{0k} \text{ for all } k,$$

and

$$\|y_n\| \to \|x_0\| = 1.$$

We can prove that $x_n \to x_0$ by applying Remark 7.

Since ℓ has property H, we clearly have

Corollary 19 WLUR is equivalent to LUR on ℓ.

Theorem 20 ℓ is LUkR if and only if ℓ is UkR or $p_n > 1$ for all n.

Proof (Case $p_1 = p_2 = \dots = p_k = p_{k+1} = 1$) Employ the same example as of the UkR case.

(Case $p_{k_0} = 1$ for some k_0 and $p_{m_n} \to 1$) Suppose $p_1 = 1$. Let $x_0 = e_1$ and $x_n = e_{m_n}$. Note that

$$\|x_0 + x_{n+1} + \dots + x_{n+k}\| \to k + 1 \text{ as } n \to \infty.$$

Hence, ℓ is not LUkR.

(Case $p_n > 1$ for all n) Let $x_0 \in S(\ell)$ and $x^n_1, \dots, x^n_k \in B(\ell)$. If $\|x_0 + x^n_1 + \dots + x^n_k\| \to k + 1$, then $\|x_0 + x^n_i\| \to 2$ for all i. By locally uniform rotundity of ℓ (Theorem 16), $\|x^n_i - x_0\| \to 0$ for all i. It is clear now that $\Delta(x_0, x^n_1, \dots, x^n_k) \to 0$ as $n \to \infty$.

Mid-point locally uniform rotundity and uniform rotundity in every direction

X is **mid - point locally uniformly rotund** (MLUR) if for any $x \in S(X)$ and $x_n, y_n \in B(X)$ with $x_n + y_n \to 2x$ imply $x_n - y_n \to 0$.

X **is uniformly rotund in every direction** (URED) if, for any $x_n, z \in X$ with $\|x_n\| \to 1$, $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ imply $z = 0$.

Theorem 21 ℓ is MLUR if and only if $p_k = 1$ for at most one k.

Proof ($\Rightarrow$) This is clear, since MLUR implies R.

($\Leftarrow$) Suppose $p_1 = 1$, $p_k > 1$ for all $k \ge 2$. Let $x_n, y_n \in B(\ell)$, $x_0 \in S(\ell)$ and $x_n + y_n \to 2x_0$. Note that $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + x_0\| \to 2$ and $\|y_n + x_0\| \to 2$. Lemma 17 implies that $x_{nk} \to x_{0k}$ and $y_{nk} \to x_{0k}$ for all $k \ge 2$. Now given any subsequence n' of n we choose a subsequence n'' of n' so that $x_{n''k} \to w_{0k}$ and $y_{n''k} \to z_{0k}$ for all $k \ge 1$, where $w_{0k} = x_{0k} = z_{0k}$ for $k \ge 2$. Note that $\frac{w_{01} + z_{01}}{2} = x_{01}$. Since $\rho(x_n) \le 1$ and $\rho(y_n) \le 1$, we must have $\rho(w_0) \le 1$ and $\rho(z_0) \le 1$. And from $\rho(x_0) = 1$ we then have $\rho(w_0) = \rho(z_0) = 1$. So $w_{01} = x_{01} = z_{01}$ as well. By Remark 7, $x_{n''} - y_{n''} \to 0$ and therefore $x_n - y_n \to 0$ as desired.

Theorem 22 ℓ is URED if and only if $p_k = 1$ for at most one k.

Proof (⇒) Follows from Theorem 2.

(⇐) Suppose $\| x_n \| \to 1$, $\| x_n + z \| \to 1$, $\| x_n + \frac{z}{2} \| \to 1$, but $z \neq 0$. Thus $z_k \neq 0$ for some k with $p_k > 1$, say k = 1. There exists $\lambda_0 > 0$ such that

$$\left| x_{n1} + \frac{z_1}{2} \right|^{p_1} = \left| \frac{x_{n1}}{2} + \frac{x_{n1} + z_1}{2} \right|^{p_1}$$

$$< \left| \frac{x_{n1}}{2} \right|^{p_1} + \left| \frac{x_{n1} + z_1}{2} \right|^{p_1} - \lambda_0$$

for all large n. For these n,

$$\rho(x_n + \frac{z}{2}) \leq \frac{1}{2}\rho(x_n) + \frac{1}{2}\rho(x_n + z) - \lambda_0 \leq 1 - \lambda_0.$$

Hence $\lim_n \rho(x_n + \frac{z}{2}) \neq 1$, a contradiction. Therefore z = 0.

Full convexity and weak uniform rotundity

For $k \geq 2$, X is **fully k - convex** (kC) if for every sequence $\{x_n\}$ in B (X) with $\| x_{n_1} + \dots + x_{n_k} \| \to k$ as $n_1, \dots, n_k \to \infty$, the sequence $\{x_n\}$ is convergent.

X is **weakly uniform rotund** (WUR) if for any $x_n, y_n \in B(X)$, $\| x_n + y_n \| \to 2$ implies $x_n - y_n \to 0$ weakly.

Theorem 23 ℓ is kC if and only if $\inf_{n \neq n_0} p_k > 1$ for some n_0.

Proof (⇒) If $p_1 = p_2 = 1$, say, consider the sequence $e_1, e_2, e_1, e_2, e_1, e_2, \dots$.

If $p_{n'} \to 1$ for some subsequence {n'}, then consider the sequence $\{e_{n'}\}$. Therefore, ℓ is not kC for each of these two cases.

(⇐) Follows from the uniform rotundity of ℓ (Theorem 15.)

Theorem 24 ℓ is WUR if and only if ℓ is UR.

Proof If $p_1 = p_2 = 1$, say, let $x_n = e_1$, $y_n = e_2$. If $p_n \to 1$, let $x_n = e_n$, $y_n = e_{n+1.}$

Drop property

X has the **drop property** (D) if for every closed set C with $C \cap B(X) = \varnothing$, there exists element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

Theorem 25 ℓ has property D if and only if $\liminf_{k \to \infty} p_k > 1$.

Proof Assume, instead of considering a subsequence, $p_k \to 1$. Let $x_1 = 2_{e_1}$, $x_2 = e_1 + \frac{1}{2} e_2$, and in general, let

$$x_n = \frac{1}{2^{n-2}} e_1 + \sum_{k=1}^{n-1} \frac{1}{2^k} e_{n+1-k}.$$

Put $C = \{x_n : n \geq 1\}$. It is clear that C is a closed set, $C \cap B(\ell) = \varnothing$, and for each n, since $x_{n+1} = \frac{1}{2} x_n + \frac{1}{2} e_{n+1}$, we see that $x_k \in D(x_n, B(\ell))$ $(k \geq n)$. Thus ℓ does not have property D in this case. For the other case, ℓ is NUC by Theorem 13, and hence has property D.

Remark 26 Observe that, for the set C in the proof above, $\inf_{x \in C} \| x \| = 1$. In fact, for any Banach space X, any closed set C which does not overlap with B (X), if for some $x \in C$,

$$\inf \{ \|y\| : y \in D(x, B(X)) \cap C\} > 1,$$

then we must have

$$A(y) := D(y, B(X)) \cap C = \{y\}$$

for some $y \in C$.

Proof Suppose there is no such point $y \in C$ for some point $x_0 \in C$. Put $\alpha_0 = \inf_{y \in A(x_0)} \| y \| > 1$. Take $x_1 \in A(x_0)$ so that $\| x_1 \| \leq \alpha_0 + 1$, and put $\alpha_1 = \inf_{y \in A(x_1)} \| y \|$. Clearly $\alpha_1 \geq \alpha_0$. By induction, we can find a sequence $\{x_n\}$ in C such that

$$X_{n+1} \in A(x_n) \text{ and } \| x_{n+1} \| \leq \alpha_n + \frac{1}{2^n}$$

where $\alpha_n = \inf_{y \in A(x_n)} \| y \|$.

Writing $x_{n+1} = r_n x_n + (1 - r_n)t_n$ as a convex combination of x_n, and some point $t_n \in B(\ell)$, we see that

$$\alpha_0 \le \alpha_{n-1} \le \alpha_n \le \| x_{n+1} \| \le r_n (\alpha_{n-1} + \frac{1}{2^{n-1}}) + (1 - r_n).$$

This implies

$$r'_n := 1 - r_n \le \frac{1}{(\alpha_0 - 1)2^{n-1}},$$

and then

$$\| x_{n+1} - xn \| \le r'_n (\| xn \| + 1) \le \frac{\| x_0 \| + 1}{(\alpha_0 - 1)2^{n-1}}.$$

The sequence $\{x_n\}$ is then a Cauchy sequence, and converges to some point $c \in C$. We show that $A(c) = \{c\}$ which is a contradiction. If $x \in A(c)$, then $\alpha_n \le \| x \| \le \| c \|$ for all n. But then $\| x \| = \| c \|$ and $x = c$ since $x \in A(c)$.

Final Remark

X is said to have **property G** if every point of S (X) is a **denting point** of B (X), that is,

$$x \notin \overline{co}\,(B(X) \setminus (x + \varepsilon B(X)))$$

for all $x \in S(X)$ and all $\varepsilon > 0$.

X is said to have **property K** if the weak topology and norm topology on S (X) are equivalent.

From the relation

$$G \Leftrightarrow K + R,$$

and the property H of ℓ, we see that property G and property R are equivalent on the Nakano sequence space ℓ.

We have seen that the boundedness of the sequence $\{p_k\}$ is required for every geometric property considered. It is interesting to see what properties having the boundedness of $\{p_k\}$ as their necessary condition.

Acknowledgement

The author would like to thank Professor R. Pluciennik and Professor Y. Cui for their helpful conversations during their short visit to Thailand. The author also wish to thank an anonymous referee for his suggestions which led to substantial improvements of this paper.

References

1. Chen S, Sun H and Wu C (1991) λ- property of Orlicz spaces, *Bull Polish Acad Sci Math*, **39**, 63-9.
2. Chen S (1996) "Geometry of Orlicz spaces," Dissertationes Math.(Rozprawy Math.), 356.
3. Cui Y, Hudzik H and Pluciennik R (1998) On some uniform convexities and smoothnesses in certain sequence spaces, *Zeit Anal Anw*, **17**, 893-905.
4. Cui Y, Pluciennik R and Wang T (1997) On property β in Orlicz spaces, *Arch Math,* **68**, 1-13.
5. Day M (1973) " Normed Linear Spaces," 3rd ed., Springer-Verlag, New York.
6. Dhompongsa S, Equivalence of the properties (β) and (NUC) of Orlicz spaces, *Comm Math* (To appear).
7. Grosse - Erdmann KG(1993) Matrix transformations between the sequence spaces of Maddox, *J Math Anal Appl*, **180**, 223-38.
8. Huff R (1980) Banach spaces which are nearly uniformly convex, *Rocky Mountain J Math*, **10**, 473-9.
9. Kutzarova DN (1990) An isomorphic characterization of property (β) of Rolewicz, *Note Mat*, **10**, 347-54.
10. Maddox IJ (1967) Spaces of strongly summable sequence spaces, *Quart J Math Oxford Ser,* (2)**18**, 345-55.
11. Maddox IJ (1968) Paranormed sequence spaces generated by infinite matrices, *Proc Cambridge Philos Soc,* **64**, 335-40.
12. Nakano H (1951) Modulared sequence spaces, *Proc Japan Acad*, **27**, 508-12.
13. Rudin W (1966) "Functional Analysis," McGraw-Hill, N Y.
14. Simons S (1965) The sequence spaces (p_k) and $m(p_k)$, *Proc London Math Soc*, **15**, 422-36.
15. Smith MA (1978) Some examples concerning rotundity in Banach spaces, *Math Ann*, **233**, 155-61.
16. Wang T, Wang B and Li M (1995) *λ-coefficient of Orlicz sequence spaces*, *Fasc Math*, **2**, 179-86.