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A Finite Element Method for Viscous Incompressible Thermal Flows

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ABSTRACT A finite element method for steady-state viscous incompressible thermal flows has been developed. The finite element equations are derived from a set of coupled nonlinear Navier-Stokes equations that consists of the conservation of mass, momentum, and energy equations. These derived finite element equations are validated by developing a corresponding finite element computer program that can be executed on standard personal computers. The developed finite element formulation has been evaluated by solving viscous incompressible thermal flows past irregular geometries with different boundary conditions.

KEYWORDS: finite element, viscous incompressible, thermal flows.

INTRODUCTION

Viscous incompressible thermal flows have been the subject of many theoretical and numerical investigation. The problem is relatively complex due to the coupling between the energy equation and the equations governing the fluid motion.[1,2] These equations constitute a set of coupled nonlinear differential equations which is difficult to solve especially with irregular flow geometries and boundary conditions. In the past, the finite difference method has been the most popular method to analyze such problems. Although the finite difference method can provide accurate solution over a wide range of the Rayleigh numbers, the method is not convenient for irregular configuration. Moreover, nonuniform meshes with fine discretization can not be used easily by the finite difference method in the region of complex flow behavior.

The finite element method is one of the numerical methods that has received popularity due to its capability for solving complex structural problems.[3,4] The method has been extended to solve problems in several other fields such as in the field of heat transfer,[5,6] electromagnetics,[7] biomechanics,[8] etc. In spite of the great success of the method in these fields, its application to fluid mechanics, particularly to viscous flows, is still under intensive research. This is due to the fact that the governing differential equations for general flow problems consist of several coupled equations which are inherently nonlinear. Accurate numerical solutions thus require a vast amount of computer time and data storage. One way to minimize the amount of computer time and data storage used is to employ an adaptive meshing technique.[9,10] The technique places small elements in the regions of large change in the solution gradients to increase solution accuracy, and at the same time, uses large elements in the other regions to reduce the computational time and computer memory.

As the first step toward accurate flow solutions using the adaptive meshing technique, this paper develops a finite element formulation suitable for analysis of general viscous incompressible thermal flow problems. The formulation evaluated in this paper will be used with the adaptive meshing technique in the future. The paper starts from the Navier-Stokes equations together with the energy equation to derive the corresponding finite element equations. The computational procedure used in the development of the computer program is described. The finite element equations derived and the computer program developed are then evaluated by examples of free convection in a square enclosure, free convection in concentric cylinders, and flow in a channel with a built-in high temperature rectangular cylinder.

GOVERNING EQUATIONS

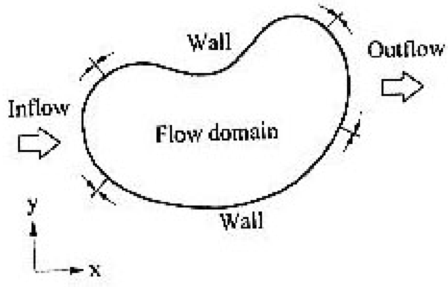
The fundamental laws used to solve the fluid flow problems are the law of: (a) conservation of mass or continuity equation, (b) conservation of momentums, and (c) conservation of energy, which constitute a set of coupled, nonlinear, partial differential equations. For laminar incompressible thermal flow, the buoyancy force is included herein as a body force in the y-momentum equation.[11] If the effects of radiation and viscous dissipation are neglected, as well as the internal heat generation is omitted, the differential equations for the two-dimensional steady-state thermal flow are,

$$u,_x + v,_y = 0 \tag{1}$$

$$\rho(uu,_x + vu,_y) = \sigma_{x,x} + \tau_{xy,y} \tag{2a}$$

$$\rho(uv,_x + vv,_y) = \tau_{xy,x} + \sigma_{y,y} - \rho g[1 - \beta(T - T_0)] \tag{2b}$$

$$\rho c(uT,_x + vT,_y) = (kT,_x),_x + (kT,_y),_y \tag{3}$$



where u and v are the velocity components in the x and y direction, respectively; T is the fluid temperature, T_0 is the reference temperature for which buoyant force vanishes, ρ is the fluid density, g is the gravitational constant, β is the volumetric coefficient of thermal expansion, c is the fluid specific heat, and k is the fluid thermal conductivity. The stress components are defined by,

$$\sigma_x = -p + 2\mu u,_x \tag{4a}$$

$$\sigma_y = -p + 2\mu v,_y \tag{4b}$$

$$\tau_{xy} = \mu(u,_y + v,_x) \tag{4c}$$

where ρ is the total pressure and μ is the fluid dynamic viscosity.

The partial differential equations, Eqs (1-3), are to be solved together with appropriate boundary conditions (see Fig 1) of,

(1) Specifying velocity components (u_i, v_i) and fluid temperature (T_i) along inflow boundary (S_i).

$$u = u_i(x,y) \tag{5a}$$

$$v = v_i(x,y) \tag{5b}$$

$$T = T_i(x,y) \tag{5c}$$

(2) Specifying surface tractions (P_x, P_y) along outflow boundary (S_0).

$$P_x = \sigma_x l + \tau_{xy} m \tag{6a}$$

$$P_y = \tau_{xy} l + \sigma_y m \tag{6b}$$

where l and m are direction cosines of the unit vector normal to the boundary.

(3) Specifying velocity components (u_w, v_w) and fluid temperature (T_w) or heat flux (q_w) that flows into or out from domain along wall boundary (S_w).

$$u = u_w(x,y) \tag{7a}$$

$$v = v_w(x,y) \tag{7b}$$

$$T = T_w(x,y) \text{ or } q = q_w(x,y) \tag{7c}$$

Finite Element Formulation

The basic unknowns for the incompressible thermal flow governing differential equations, Eqs (1-3), after substituting the stress components from Eq (4), are the velocity components, u, v, the temperature, T, and the pressure p. The six-node triangular element as suggested in Ref [12] for non-isothermal flow is used in this paper for the development of the finite element equations. The element assumes quadratic interpolation for the velocity component and the temperature distributions and linear interpolation for the pressure distribution according to their highest derivative orders in the differential Eqs (1-3) as,

$$u(x,y) = N_\alpha u_\alpha \tag{8a}$$

$$v(x,y) = N_\alpha v_\alpha \tag{8b}$$

$$T(x,y) = N_\alpha T_\alpha \tag{8c}$$

$$p(x,y) = H_\lambda p_\lambda \tag{8d}$$

where $\alpha = 1, 2, \ldots, 6$; $\lambda = 1, 2, 3$; N_α are the element interpolation functions for the velocity components and the temperature, and N_λ are the element interpolation functions for the pressure.

To derive the finite element equations, the method of weighted residuals [6] is applied to the momentum Eqs (2a-b), the energy Eq (3), and the continuity Eq (1),

$$\int_A N_\alpha\, \rho(u u,_x + v u,_y)\, dA = \int_A N_\alpha (\sigma_{x,x} + \tau_{xy,y})\, dA \quad (9a)$$

$$\int_A N_\alpha\, \rho(u v,_x + v v,_y)\, dA = \int_A N_\alpha (\tau_{xy,x} + \sigma_{y,y})\, dA - \int_A N_\alpha\, \rho g\,[1 - \beta(T - T_0)]\, dA \quad (9b)$$

$$\int_A N_\alpha\, \rho c(u T,_x + v T,_y)\, dA = \int_A N_\alpha [(k T,_x),_x + (k T,_y),_y]\, dA \quad (9c)$$

$$\int_A H_\lambda (u,_x + v,_y)\, dA = 0 \quad (9d)$$

where A is the element area. For the six-node triangular element, Eqs (9a-d) consist of 21 finite element equations. Gauss's theorem is then applied to Eqs (9a-c) to generate the boundary integral terms associated with the surface tractions and heat flux. With the use of Eqs (5-7), Eqs (9a-c) become,

$$\int_A N_\alpha\, \rho(u u,_x + v u,_y)\, dA + \int_A (N_{\alpha,x}\, \sigma_x + N_{\alpha,y}\, \tau_{xy})\, dA = \int_{S_o} N_\alpha\, P_x\, dS_o \quad (10a)$$

$$\int_A N_\alpha\, \rho(u v,_x + v v,_y)\, dA + \int_A (N_{\alpha,x}\, \tau_{xy} + N_{\alpha,y}\, \sigma_y)\, dA = \int_{S_o} N_\alpha\, P_y\, dS_o - \int_A N\alpha\, \rho\, g\,[1 - \beta(T - T_0)]\, dA \quad (10b)$$

$$\int_A N_\alpha\, \rho c(u T,_x + v T,_y)\, dA + \int_A N_{\alpha,x}(k T,_x)\, dA + \int_A N_{\alpha,y}(k T,_y)\, dA = \int_{S_w} N_\alpha\, q_s\, dS_w \quad (10c)$$

Substituting the element velocity component distributions, the temperature distribution, and the pressure distribution from Eqs (8a-d), and the stress components from Eqs (4a-c), the finite element equations can be written in the form,

$$K_{\alpha\beta\gamma^x}\, u_\beta\, u_\gamma + K_{\alpha\beta\gamma^y}\, v_\beta\, u_\gamma - H_{\alpha\lambda^x}\, p_\lambda + S_{\alpha\beta^{xx}}\, u_\beta + S_{\alpha\beta^{xy}}\, v_\beta = Q_{\alpha^u} \quad (11a)$$

$$K_{\alpha\beta\gamma^x}\, u_\beta\, v_\gamma + K_{\alpha\beta\gamma^y}\, v_\beta\, v_\gamma - H_{\alpha\lambda^y}\, p_\lambda - K_{\alpha\beta}\, T_\beta + S_{\alpha\beta^{yx}}\, u_\beta + S_{\alpha\beta^{yy}}\, v_\beta = Q_{\alpha^v} - C_\alpha - D_\alpha \quad (11b)$$

$$K_{\alpha\beta\gamma^x}\, u_\beta\, T_\gamma + K_{\alpha\beta\gamma^y}\, v_\beta\, T_\gamma + M_{\alpha\beta^{xx}}\, T_\beta + M_{\alpha\beta^{yy}}\, T_\beta = Q_{\alpha^T} \quad (11c)$$

$$H_{\beta\mu^x}\, u_\beta + H_{\beta\mu^y}\, v_\beta = 0 \quad (11d)$$

where the coefficients in element matrices are in the form of the integrals over the element area and along the element edges S_o and S_w as,

$$K_{\alpha\beta\gamma^x} = \int_A N_\alpha\, N_\beta\, N_{\gamma,x}\, dA \quad (12a)$$

$$K_{\alpha\beta\gamma^y} = \int_A N_\alpha\, N_\beta\, N_{\gamma,y}\, dA \quad (12b)$$

$$H_{\alpha\lambda^x} = \frac{1}{\rho}\int_A N_{\alpha,x}\, H_\lambda\, dA \quad (12c)$$

$$H_{\alpha\lambda^y} = \frac{1}{\rho}\int_A N_{\alpha,y}\, H_\lambda\, dA \quad (12d)$$

$$S_{\alpha\beta^{xx}} = 2\nu\int_A N_{\alpha,x}\, N_{\beta,x}\, dA + \nu\int_A N_{\alpha,y}\, N_{\beta,y}\, dA \quad (12e)$$

$$S_{\alpha\beta^{yy}} = \nu\int_A N_{\alpha,x}\, N_{\beta,x}\, dA + 2\nu\int_A N_{\alpha,y}\, N_{\beta,y}\, dA \quad (12f)$$

$$S_{\alpha\beta^{xy}} = \nu\int_A N_{\alpha,y}\, N_{\beta,x}\, dA \quad (12g)$$

$$S_{\alpha\beta^{yx}} = \nu\int_A N_{\alpha,x}\, N_{\beta,y}\, dA \quad (12h)$$

$$M_{\alpha\beta^{xx}} = \frac{k}{\rho c}\int_A N_{\alpha,x}\, N_{\beta,x}\, dA \quad (12i)$$

$$M_{\alpha\beta^{yy}} = \frac{k}{\rho c}\int_A N_{\alpha,y}\, N_{\beta,y}\, dA \quad (12j)$$

$$K_{\alpha\beta} = g\,\beta\int_A N_\alpha\, N_\beta\, dA \quad (12k)$$

$$Q_{\alpha^u} = \frac{1}{\rho}\int_{S_o} N_\alpha\, P_x\, dS_o \quad (12l)$$

$$Q_{\alpha^v} = \frac{1}{\rho}\int_{S_o} N_\alpha\, P_y\, dS_o \quad (12m)$$

$$Q_{\alpha^T} = \frac{1}{\rho c}\int_{S_w} N_\alpha\, q_s\, dS_w \quad (12n)$$

$$C_\alpha = g\int_A N_\alpha\, dA \quad (12o)$$

$$D_\alpha = g\,\beta\, T_0\int_A N_a\, dA \quad (12p)$$

$$H_{\beta\mu^x} = \frac{1}{\rho}\int_A N_{\beta,x}\, H_\mu \, dA \qquad (12q)$$

$$H_{\beta\mu^y} = \frac{1}{\rho}\int_A N_{\beta,y}\, H_\mu \, dA \qquad (12r)$$

where ν is the fluid kinematic viscosity defined by,

$$\nu = \frac{\mu}{\rho} \qquad (13)$$

These element matrices are evaluated in closed-form ready for computer programming. Details of the derivation for these element matrices are omitted herein for brevity.

Computational Procedure

The derived finite element equations, Eqs (11a-d), are nonlinear. These nonlinear algebraic equations are solved by applying the Newton-Raphson iteration technique [13] by first writing the unbalanced values from the set of the finite element Eqs (11a-d) as,

$$F_{\alpha^u} = K_{\alpha\beta\gamma^x}\, u_\beta u_\gamma + K_{\alpha\beta\gamma^y}\, v_\beta u_\gamma - H_{\alpha\lambda^x}\, p_\lambda + S_{\alpha\beta^{xx}}\, u_\beta + S_{\alpha\beta^{xy}}\, v_\beta - Q_{\alpha^u} \qquad (14a)$$

$$F_{\alpha^v} = K_{\alpha\beta\gamma^x}\, u_\beta v_\gamma + K_{\alpha\beta\gamma^y}\, v_\beta v_\gamma - H_{\alpha\lambda^y}\, p_\lambda + S_{\alpha\beta^{yx}}\, u_\beta + S_{\alpha\beta^{yy}}\, v_\beta - K_{\alpha\beta} T_\beta - Q_{\alpha^v} + C_\alpha + D_\alpha \qquad (14b)$$

$$F_{\alpha^T} = K_{\alpha\beta\gamma^x}\, u_\beta T_\gamma + K_{\alpha\beta\gamma^y}\, v_\beta T_\gamma + M_{\alpha\beta^{xx}}\, T_\beta + M_{\alpha\beta^{yy}}\, T_\beta - Q_{\alpha^T} \qquad (14c)$$

$$F_{\beta^p} = H_{\beta\mu^x}\, u_\beta + H_{\beta\mu^y}\, v_\beta \qquad (14d)$$

This leads to a set of algebraic equations with the incremental unknowns of the element nodal velocity components, temperatures, and pressures in the form,

$$\begin{bmatrix} K_{uu} & K_{uv} & 0 & K_{up} \\ K_{vu} & K_{vv} & K_{vT} & K_{vp} \\ K_{Tu} & K_{Tv} & K_{TT} & 0 \\ K_{pu} & K_{pv} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta u_\beta \\ \Delta v_\beta \\ \Delta T_\beta \\ \Delta p_\lambda \end{Bmatrix} = -\begin{Bmatrix} F_{\alpha^u} \\ F_{\alpha^v} \\ F_{\alpha^T} \\ F_{\beta^p} \end{Bmatrix} \qquad (15)$$

where

$$K_{uu} = K_{\alpha\beta\gamma^x}\, u_\gamma + K_{\alpha\gamma\beta^x}\, u_\gamma + K_{\alpha\gamma\beta^y}\, v_\gamma + S_{\alpha\beta^{xx}} \qquad (16a)$$

$$K_{uv} = K_{\alpha\beta\gamma^y}\, u_\gamma + S_{\alpha\beta^{xy}} \qquad (16b)$$

$$K_{up} = -H_{\alpha\lambda^x} \qquad (16c)$$

$$K_{vu} = K_{\alpha\beta\gamma^x}\, v_\gamma + S_{\alpha\beta^{yx}} \qquad (16d)$$

$$K_{vv} = K_{\alpha\gamma\beta^x}\, u_\gamma + K_{\alpha\beta\gamma^y}\, v_\gamma + K_{\alpha\gamma\beta^y}\, v_\gamma + S_{\alpha\beta^{yy}} \qquad (16e)$$

$$K_{vT} = -K_{\alpha\beta} \qquad (16f)$$

$$K_{vp} = -H_{\alpha\lambda^y} \qquad (16g)$$

$$K_{Tu} = K_{\alpha\beta\gamma^x}\, T_\gamma \qquad (16h)$$

$$K_{Tv} = K_{\alpha\beta\gamma^y}\, T_\gamma \qquad (16i)$$

$$K_{TT} = K_{\alpha\gamma\beta^x}\, u_\gamma + K_{\alpha\gamma\beta^y}\, v_\gamma + M_{\alpha\beta^{xx}} + M_{\alpha\beta^{yy}} \qquad (16j)$$

$$K_{pu} = H_{\beta\mu^x} \qquad (16k)$$

$$K_{pv} = H_{\beta\mu^y} \qquad (16l)$$

In the above Eqs (16a-l), u_γ and v_γ are the values of the velocity components and T_γ is the values of the temperature at the i^{th} iteration. The iteration process is terminated if the percentage of the overall change compared to the previous iteration is less than the specified value.

The final form of the finite element equations, Eqs (15-16), and the iteration procedure described are used in the development of a finite element computer program that can be executed on standard personal computers. The program has been verified by solving a number of examples that have known results before applying to solve more complex flow problems. Selected examples are presented in the next section.

Examples

In order to evaluate the finite element formulation and demonstrate its capability, three examples of viscous incompressible thermal flows are presented herein.

Free convection in a square enclosure

The first example for validating the finite element formulation developed is the problem of free convection in a square enclosure. The square enclosure of side L, shown in Fig 2, is bounded by the two vertical plates with specified temperatures

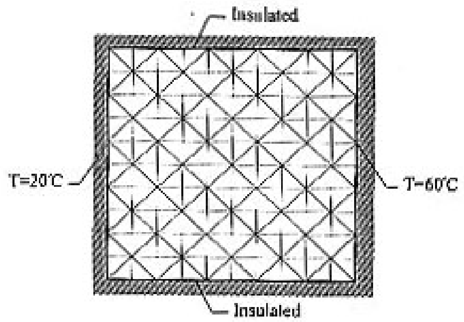
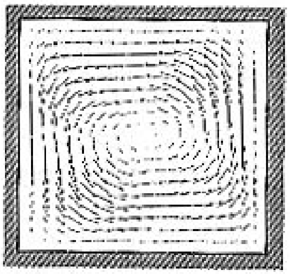
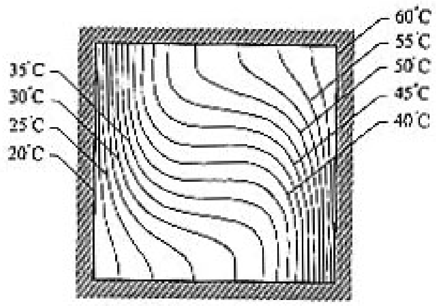


Fig 2. Finite element model and boundary conditions of free convection in a square enclosure.

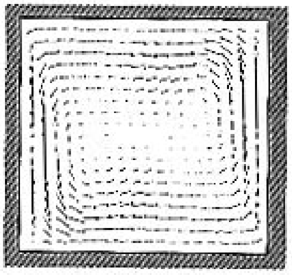


(a) Velocity distribution.

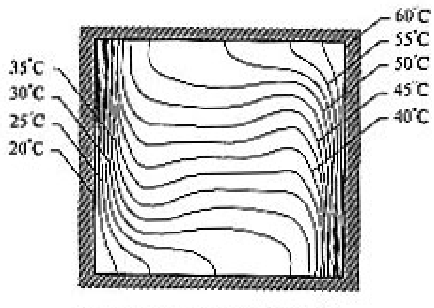


(b) Temperature distribution.

Fig 3. Flow solutions for free convection in a square enclosure with Ra = 10^4.

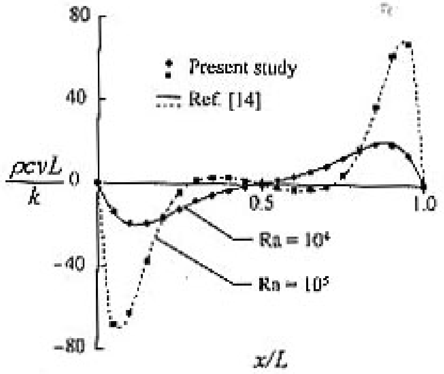


(a) Velocity distribution.

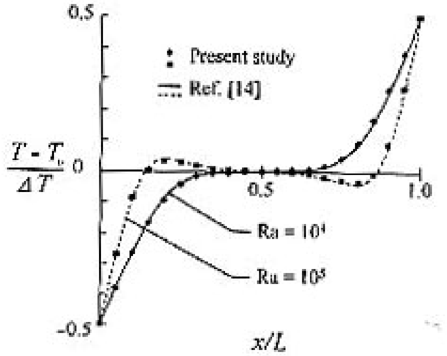


(b) Temperature distribution.

Fig 4. Flow solutions for free convection in a square enclosure with Ra = 10^5.



(a) Velocity profiles.



(b) Temperature distributions.

Fig 5. Comparative flow solutions along the mid-height of the square enclosure.

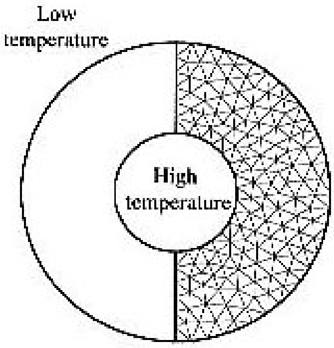
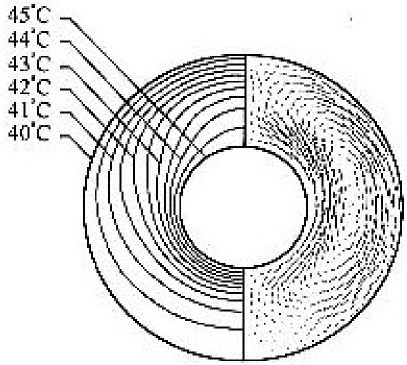
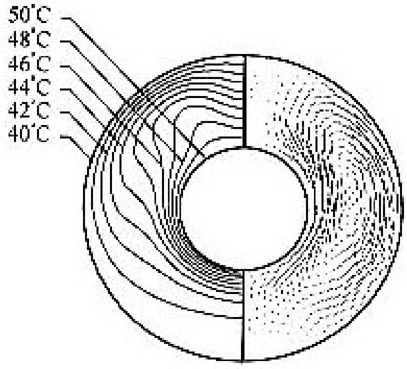


Fig 6. Finite element model of free convection in concentric cylinders.

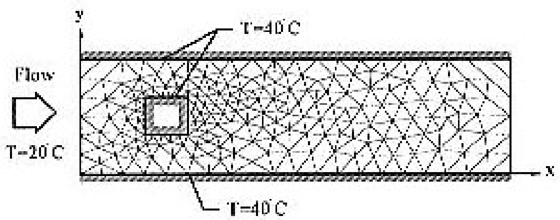


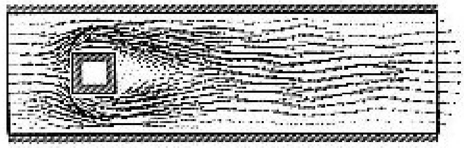
(a) Ra = 2500.



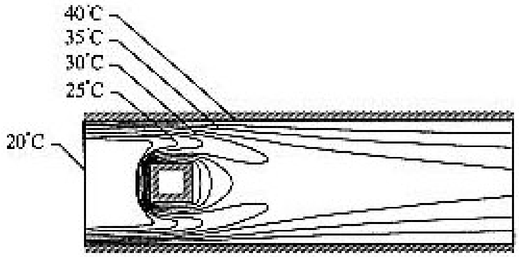
(b) Ra = 5000.

Fig 7. Predicted velocity vectors and isotherms of free convection in concentric cylinders.

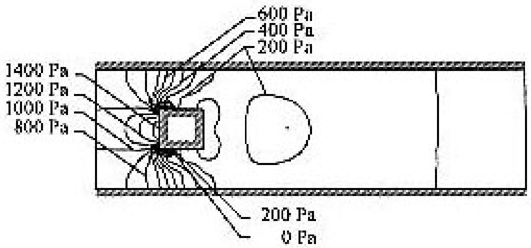




(a) Predicted velocity vectors.

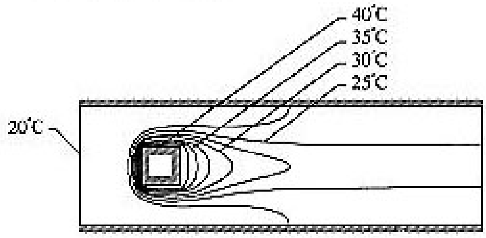


(b) Predicted temperature distributions.

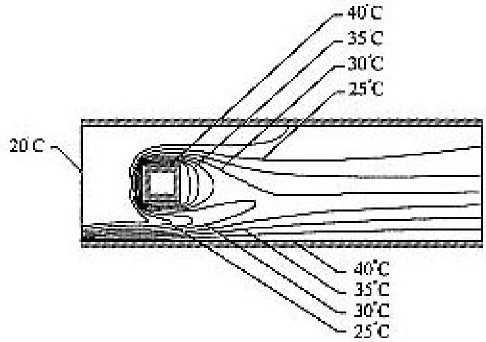


(C) Predicted pressure distributions.

Fig 9. Flow solutions in a channel with built-in high temperature rectangular cylinder.



(a) Both wall insulated.



(b) Only top wall insulated.

of 20°C and 60°C and insulated plates on the top and bottom surfaces. The problem was analyzed by other technique[14] for which the result can be used for comparison. The finite element model consisting of 200 elements and 441 nodes, as shown in Fig 2, is used in this study. Fig 3(a-b) show the predicted velocity and temperature distributions, respectively, for the case with the Rayleigh number of 10^4. The figures show relatively smooth flow velocity vectors that circulate in the counterclockwise direction and the smooth temperature distribution. The same analysis is repeated but with the Rayleigh number of 10^5. Different flow patterns of the velocity and temperature distributions are obtained as shown in Figs 4(a-b). The flow velocity pattern indicates two regions of circulation, both in counterclockwise direction. The predicted velocity profiles and the temperature distributions along the mid-height of the square enclosure for both flow cases are compared with the results from Ref 14 as shown in Figs 5(a-b), respectively. The figures show good agreement of the solutions for both the flow cases.

Free convection in concentric cylinders

The second example is the problem of free convection in the annular space between long, horizontal concentric cylinders. The finite element model consisting of 271 elements and 600 nodes is shown in Fig 6. The concentric cylinders has high temperature on the inner cylinder and lower temperature on the outer cylinder. Two analysis cases with the temperature difference between the inner and outer cylinders of 5°C and 10°C (Rayleigh numbers of 2500 and 5000) are performed. Fig 7(a) shows the predicted fluid temperature distribution and velocity vectors for the case with the Rayleigh number of 2500. The figure shows smooth temperature distribution and flow velocity pattern that circulates in the clockwise direction. However, for the case with higher Rayleigh number of 5000, the flow velocity pattern has changed to two regions of circulation with a small region of counterclockwise flow pattern appears on the upper portion of the cylinders as shown in Fig 7(b). These results highlight a more complex flow behavior that can occur from a slight change of the boundary condition.

Flow in a channel with a built-in high temperature rectangular cylinder

To further demonstrate the capability of the finite element formulation developed for a more complex problem, the problem of viscous thermal flow through a channel with a built-in high temperature rectangular cylinder as shown in Fig 8 is selected. The flow at temperature of 20°C and Reynolds number of 100 enters through the left boundary of a channel that has temperatures on both the upper and lower walls of 40°C. The fluid flows past the built-in rectangular cylinder that also has higher temperature of 40°C. The finite element model used for the analysis, as shown in the figure, consists of 390 elements and 852 nodes.

The predicted flow solutions are shown in Figs 9(a-c). Fig 9(a) shows the velocity pattern of the fluid that flows past the rectangular cylinder with two small regions of circulation occuring behind it. Fig 9(b) shows the predicted temperature contours with increased fluid temperature near both of the walls and the rectangular cylinder. Fig 9(c) shows the predicted flow pressure distribution that causes the fluid to flow from the left to the right of the channel.

To demonstrate that the finite element formulation can handle different boundary conditions easily, the analysis is repeated for the case when both the upper and lower walls are insulated. Fig 10(a) shows the predicted temperature contours indicating that the fluid temperature increases as it flows past the high temperature rectangular cylinder. Fig 10(b) also shows the predicted temperature contours for the case when the upper wall is insulated and the lower wall temperature is maintained at 40°C. These results demonstrate the capability of the finite element formulation developed that can provide insight into the complexity of the viscous thermal flow behaviors under different boundary conditions.

Concluding Remarks

A finite element method for steady-state viscous incompressible thermal flow is presented. The finite element equations were derived from the governing flow equations that consist of the conservation of mass, momentum, and energy equations. The derived finite element equations are nonlinear requiring an iterative technique solver. The Newton-Raphson iteration method is applied to solve these nonlinear equations for solutions of the nodal velocity components, temperatures, and pressures. The corresponding computer program that can be executed on standard personal computers has been developed. The finite element formulation developed have been verified and evaluated by the three examples of free convection in a square enclosure, free convection in concentric cylinders, and flow in a channel with a built-in high temperature rectangular cylinder. The examples demonstrate the capability of the finite element formulation that can provide insight to complex viscous thermal flow behaviors.

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