SHORT REPORTS

A NOTE ON THE EXISTENCE OF INTERFACIAL WAVES OVER AN OBSTACLE

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ABSTRACT

We investigate the existence of solutions of the steady forced modified KdV equation with forcing term (SFMKdV) $\eta_{xxx} = -A\eta^2\eta_x + B\eta_x + Cb_x$. This type of equation appears in the context of two-dimensional steady capillary-gravity waves on the interface of two immiscible fluids over a small obstruction. It is shown that there exist both symmetric and unsymmetric solutions to this equation if $B > 0$, i.e. when the flow is in the subcritical regime.

1. INTRODUCTION

In the study of symmetric waves of a two-layer fluid over a small obstruction of compact support at the rigid bottom, Choi and Asavanant [1] derived the steady forced modified KdV equation (SFMKdV), using a unified asymptotic method, in the following form

$$\eta_{xxx} = -A\eta^2\eta_x + B\eta_x + Cb_x, \tag{1}$$

Here $\eta(x)$ denotes vertical position of the interface between the two fluids, A, B, and C are constants. In particular, A and C are always positive. The function $b(x)$ represents the shape of the bottom obstuction with compact support. All the subscripts denote derivatives with respect to x. Existence theorem for symmetric solutions was given and this was confirmed by the numerical calculations. Mielke [2], Shen [3], and Sun and Shen [4] proved the validity of the asymptotic theory and justified its use for this type of fluid-flow problems.

For the two-layer fluid system, it was shown that the signature of B distinguishes the characteristics of the fluid flow. That is the flow is subcritical when $B > 0$ and is supercritical when $B < 0$. We shall restrict the analysis to subcritical flow regime ($B > 0$). In this paper we give detail proofs of existence theorems of both symmetric and unsymmetric solutions of (1).

2. SYMMETRIC SOLUIONS

Following Choi and Asavanant [1], we look for a solution $\eta(x)$ such that $B > 0$ and

$$\lim_{|x|\to\infty} (d/dx)^j \eta(x) = 0 \qquad j = 0,1,2.$$

Integrating (1) from $-\infty$ to x, it follows that

$$B\eta - \eta_{xx} = \frac{A}{3}\eta^3 + b_1(x) \tag{2}$$

where $b_1(x) = -Cb(x)$. It can easily be shown that (2) is equivalent to an integral equation

$$\eta(x) = \int_{-\infty}^{\infty} K(x,\xi)\left(\frac{A}{3}\eta^3(\xi) + b_1(\xi)\right)d\xi.$$

Here $K(x,\xi) = e^{-\sqrt{B}|x-\xi|}/2\sqrt{B}$ is the Green's function which is a solution of

$$BK(x,\xi) - K_{xx}(x,\xi) = \delta(x-\xi), \quad -\infty < x,\xi < \infty.$$

We now define

$$T(\eta) = \int_{-\infty}^{\infty} K(x,\xi)\left(\frac{A}{3}\eta^3(\xi) + b_1(\xi)\right)d\xi$$

$$\|u\| = \|u\|_\infty = \sup_{x\in\Re}|\eta(x)|$$

$$H = \left\{u \,\middle|\, u \in C(\Re), \left\|e^{\sqrt{B}|x|}u\right\| < \infty\right\}.$$

Clearly, H is a metric space and is complete. We give another definition

$$B_M = \left\{u \,\middle|\, u \in H, \|u\| \le M, 0 < M < \infty\right\}.$$

Lemma 1. $\|T(\eta)\| \le M$ for $\eta \in B_M$ *if* $\frac{AM^2}{3} + \frac{\|b_1\|}{M} \le B.$

Proof.

$$\begin{aligned}\|T(\eta)\| &= \sup_{x\in\Re}\left|\int_{-\infty}^{\infty} K(x,\xi)\left(\frac{A}{3}\eta^3(\xi) + b_1(\xi)\right)d\xi\right| \\ &\le \left\|\frac{A}{3}\eta^3 + b_1\right\| \sup_{x\in\Re}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{B}}\exp\left(-\sqrt{B}|x-\xi|\right)d\xi \\ &\le \left(\frac{AM^3}{3} + \|b_1\|\right)\Big/B \\ &\le M\end{aligned}$$

as required.

Next we want to prove that $T(\eta)$ decays rapidly so that we may consider the behavior of $e^{\sqrt{B}|x|}|T(\eta)(x)|$ when $|x|$ is large.

Lemma 2. $\sup_{x\in\Re}\exp\left(\sqrt{B}|x|\right)|T(\eta)(x)| < \infty$ *for* $\eta \in B_M$.

Proof. It suffices to prove the case when x > 0.

$$\begin{aligned}e^{\sqrt{B}|x|}|T(\eta)(x)| = & \left|\int_{-\infty}^{\infty}\exp\left(\sqrt{B}x - \sqrt{B}|x-\xi|\right)\left(\frac{A}{3}\eta^3(\xi) + b_1(\xi)\right)d\xi\right| \; 2\sqrt{B} \\ = & \left|\int_{-\infty}^{x}\exp\left(\sqrt{B}\xi\right)\left(\frac{A}{3}\eta^3(\xi) + b_1(\xi)\right)d\xi\right. \\ & \left. + \int_{x}^{\infty}\exp\left(\sqrt{B}(2x-\xi)\right)\left(\frac{A}{3}\eta^3(\xi) + b_1(\xi)\right)d\xi\right| \; 2\sqrt{B} \\ = & \left|\int_{-\infty}^{x}\frac{A}{3}\exp\left(\sqrt{B}\xi - 3\sqrt{B}|\xi|\right)\left(\eta(\xi)\exp\left(\sqrt{B}\right)|\xi|\right)^3 - \exp\left(\sqrt{B}\xi\right)b_1(\xi)d\xi\right.\end{aligned}$$

$$+ \quad \int_x^\infty \frac{A}{3}\exp\left(\sqrt{B}(2x-\xi)-3\sqrt{B}|\xi|\right)\left(\eta(\xi)\exp\left(\sqrt{B}|\xi|\right)\right)^3$$

$$+ \quad b_1(\xi)\exp\left(\sqrt{B}(2x-\xi)\right)d\xi\Big| \; 2\sqrt{B}$$

$$\leq \quad \left|\sup_{x\in\Re}\left(\eta(x)\exp\left(\sqrt{B}|x|\right)\right)^3 \int_{-\infty}^{x}\exp\left(-\sqrt{B}|\xi|\right)d\xi\right.$$

$$+ \quad \int_x^\infty \exp\left(\sqrt{B}(2x-3\xi)\right)d\xi\left|\frac{A}{6\sqrt{B}}\right.$$

$$+\left|\int_{-\infty}^{\infty}\exp\left(\sqrt{B}\xi\right)b_1(\xi)d\xi+\int_x^\infty b_1(\xi)\exp\left(\sqrt{B}(2x-\xi)\right)d\xi\right| \; 2\sqrt{B}$$

$$\leq \quad \exp\left(-\sqrt{B}x\right)\sup_{x\in\Re}\left(\eta(x)\exp\left(\sqrt{B}|x|\right)\right)^3 \Big/ 6B$$

$$+ \quad \int_{\mathrm{supp}(b)} N\exp\left(\sqrt{B}\xi\right)d\xi \Big/ 2\sqrt{B}$$

$$< \quad \infty.$$

Since $\eta\in H$, where $N=\max_{\xi\in\Re}|b_1(\xi)|$. Hence,

$$\sup_{x>0}\exp\left(\sqrt{B}x\right)|T(\eta)(x)|<\infty.$$

Similarly, one can easily show that

$$\sup_{x<0}\exp\left(-\sqrt{B}x\right)|T(\eta)(x)|<\infty.$$

This completes the proof. Now we shall state the existence theorem for symmetric solutions of (2).

Theorem 1. $B\eta-\eta_{xx}=\frac{A}{3}\eta^3+b_1(x), -\infty<x<\infty$ has a solution which decays exponentially at $|x|=\infty$ if B is sufficiently large.

Proof.

$$\|T(\eta_1)-T(\eta_2)\| \;\leq\; \sup_{x\in\Re}\left|\frac{A}{3}\int_{-\infty}^{\infty}K(x,\xi)\left(\eta_1^3(\xi)-\eta_2^3(\xi)\right)d\xi\right|$$

$$\leq \; \sup_{x\in\Re}\frac{A}{3}\int_{-\infty}^{\infty}K(x,\xi)\left|\eta_1^2+\eta_1\eta_2+\eta_2^2\right||\eta_1-\eta_2|d\xi$$

$$\leq \; AM^2\|\eta_1-\eta_2\|/B$$

$$=\frac{AM^2}{B}\|\eta_1-\eta_2\|.$$

Hence we can see from Lemmas 1 and 2 that T is a contraction mapping if $B>\max\left\{\left(\frac{AM^2}{3}+\frac{\|b_1\|}{M}\right)^{2/3},\left(AM^2\right)^{2/3}\right\}$, and the integral equation $\eta=T(\eta)$ has the unique solution in B_M. Now

$$\begin{aligned}\eta_{xx}(x) &= \int_{-\infty}^{\infty} K_{xx}(x,\xi)\left(\frac{A}{3}\eta^3(\xi)+b_1(\xi)\right)d\xi \\ &= \int_{-\infty}^{\infty} B(x,\xi)\left(\frac{A}{3}\eta^3(\xi)+b_1(\xi)\right)d\xi - \frac{A}{3}\eta^3(x)-b_1(x) \\ &= B\eta(x)-\frac{A}{3}\eta^3(x)-b_1(x),\end{aligned}$$

where $BK(x,\xi)-K_{xx}(x,\xi)=\delta(x-\xi)$. Hence $\eta\in C^2(\Re)$, and it follows from the right hand side of the above equation that $\eta\in C^3(\Re)$.

3. UNSYMMETRIC SOLUTIONS

Next we attempt to find periodic solutions of (2) when $b(x)=0$. Assume that $\eta(x)$ and $\eta_x(x)$ are given at some point $x=x_0$ and $\eta(x_0)=\alpha, \eta_x(x_0)=\beta$. We multiply η_x to (2) and integrate the resulting equation from x_0 to $x>x_0$. This yields

$$(\eta_x(x))^2 = -\frac{A}{6}\eta^4 + B\eta^2 + d = f(\eta),$$

where $d=\beta^2+\frac{A}{6}\alpha^4-B\alpha^2$. To find the solution of this equation, we consider the cases $d>0, d=0$ and $d<0$ separately. If $d>0, f(\eta)$ can be factored as $\left(-\frac{A}{6}\right)(\eta^2-\xi_0)(\eta^2-\xi_1)$. Here $\xi_1<0<\xi_0$, and

$$\xi_0 = \frac{3}{A}\left(B+\left(B^2+\frac{2Ad}{3}\right)^{1/2}\right)$$

$$\xi_1 = \frac{3}{A}\left(B-\left(B^2+\frac{2Ad}{3}\right)^{1/2}\right).$$

The solution is then $\eta=\xi_0^{1/2}\cos\phi$, where

$$\gamma(x-x_0)=\int_{\phi_0}^{\phi}\left(1-k^2\sin^2\theta\right)^{-1/2}d\theta,$$

$$\phi_0=\cos^{-1}\left(\alpha\xi_0^{-1/2}\right),$$

$$\gamma=\left(\frac{A(\xi_0-\xi_1)}{6}\right)^{1/2},$$

$$k^2=\frac{\xi_0}{\xi_0-\xi_1}<1.$$

It is clear that $\frac{dx}{d\phi}>0$. Hence $\eta(x)$ intersects the x - axis repeatedly. Suppose $\{x_i\}$ is the set of points where $\eta(x_i)=0$ for all i and $x_0\le x_1<x_2<x_3<\ldots$. Then by assuming x_i as the corresponding point of $2n\pi+\frac{\pi}{2}$ for some $n\in\mathbf{Z}$,

$$\int_{x_i}^{x_{i+2}}\eta(x)dx = \int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{5\pi}{2}} \xi_0^{1/2}\cos\phi\left(\frac{dx}{d\phi}\right)d\phi \qquad \text{for some } n\in\mathbf{Z}$$

$$= \int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{5\pi}{2}} \xi_0^{1/2}\gamma^{-1}\cos\phi\left(1-k^2\sin^2\phi\right)^{-1/2}d\phi$$

$$= \xi_0^{1/2}\gamma^{-1}\int_0^{2\pi}\sin\phi\left(1-k^2\cos^2\phi\right)^{-1/2}d\phi$$

$$= \xi_0^{1/2}\gamma^{-1}\int_{-\pi}^{\pi}\sin\phi\left(1-k^2\cos^2\phi\right)^{-1/2}d\phi$$

$$= 0.$$

This implies that the mean value of this solution $\eta(x)$ over one period is zero. If $d = 0$, (2) has a solitary wave solution. If $d < 0, B^2 + \frac{2Ad}{3} > 0$ and $f(\eta) = 0$ has two distinct roots. In this case, we multiply $4\eta^2$ to (2). This yields

$$\left(v_x\right)^2 = -\frac{2A}{3}v^3 + 4Bv^2 + 4dv = g(v) \qquad (3)$$

with $v = \eta^2$

The condition $f(\eta) = 0$ gives three different roots

$$\xi_0 = 3B + 3\sqrt{B^2 + \frac{2Ad}{3}}$$

$$\xi_1 = 3B - 3\sqrt{B^2 + \frac{2Ad}{3}}$$

$$\xi_2 = 0,$$

where $\xi_0 > \xi_1 > \xi_2$. We can now express solution of (3) as

$$v = \xi_0\cos^2\phi + \xi_1\sin^2\phi,$$

where

$$\alpha^{1/2}x = \int_0^{\phi}\left(1-\beta^2\sin^2\theta\right)^{-1/2}d\theta,$$

$$\alpha = \xi_0 - \xi_2 > 0,$$

$$\beta^2 = \left(\frac{\xi_0 - \xi_1}{\xi_0 - \xi_2}\right) < 1.$$

It follows that $\eta = \pm\left(\xi_0\cos^2\phi + \xi_1\sin^2\phi\right)^{1/2}$. If $B^2 + \frac{2Ad}{3} = 0$, we have

$$\left(v_x\right)^2 = -Av\left(v - \frac{3B}{A}\right)^2.$$

Therefore the only possibilities are $v = 0$ or $v = 3B/A$, i.e., $\eta(x) = 0$ or $\eta = \pm\left(\frac{3B}{A}\right)^{1/2}$. If $B^2 + \frac{2Ad}{3} < 0$,

$$\left(v_x\right)^2 = -Av\left(\gamma^2 + (v-\delta)^2\right) \quad \text{for some } \gamma, \delta.$$

Hence only $\eta = 0$ is possible. These show that we can find all solutions analytically for (2) with $b(x) = 0$. When $x \to -\infty$, we assume that η tends to 0. Thus only $\eta = 0$ or $d = 0$, which corresponds to solitary wave, is possible.

Now we need to know the existence of the solution of (2) in $x_- \le x \le x_+$ when $b(x) \ne 0$. In the following, we show that for some initial values of a solution at $x = x_-$, the solution always exists in $[x_-, x_+]$ and is a C^2 – function.

Theorem 2. $\eta_{xx} = -\frac{A}{3}\eta^3 + B\eta - b_1(x),\ A,B > 0$ with initial data $\eta(x_-) = \alpha$ and $\eta_x(x_-) = \beta$ has a C^2 – solution for $x_- \le x \le x_+$.

Proof. It suffices to show that η is bounded. Without loss of generality, we can assume $x_- = -1$ and $x_+ = 1$. Multiplying η_x to the given equation and integrating it from -1 to x, yields

$$\begin{aligned}(\eta_x)^2 &= -\frac{A}{6}\eta^4(x) + B\eta^2(x) + (\eta_x(-1))^2 + \frac{A}{6}(\eta(-1))^4 \\ &\quad -B(\eta(-1))^2 - 2\int_{-1}^{x} b_1(t)\eta'(t)dt \\ &= -\frac{A}{6}\left(\eta^2 - \frac{3B}{A}\right)^2 + \beta^2 + \frac{A}{6}\alpha^4 - B\alpha^2 \\ &\quad + \frac{3B^2}{2A} - 2\int_{-1}^{x} b_1(t)\eta'(t)dt.\end{aligned}$$

Hence

$$\begin{aligned}(\eta_x)^2 &\le N + 2\int_{-1}^{x} |b_1(t)\eta'(t)|dt \\ &\le N + 2\int_{-1}^{x} 8|b_1(t)|^2 + \frac{|\eta'(t)|^2}{8}dt \qquad (4)\end{aligned}$$

by Young's inequality, when $N = \beta^2 + \frac{A}{6}\alpha^4 - B\alpha^2 + \frac{3}{2A}B^2$ and $(\ )' = \frac{d}{dt}$. Suppose that η is not bounded in $[-1,1]$, then there exists a point $x_0 \in [-1,1]$ such that $|\eta| \to \infty$ as $x \to x_0$. Then $x_0 > -1 + \delta$ for some $\varepsilon > 0$ by the existence theorem in ordinary differential equation. Let $x_0 = \inf\left\{\xi \in [-1,1] \mid \lim_{x \to \xi^-} \eta(x) = \infty\right\}$. We choose δ so that $-1 < \delta < x_0$. Then the solution of the given differential equation exists in

$[-1,\delta]$, and by (4), $(\eta_x)^2 \le \frac{1}{8}(\delta + 1) \sup_{-1\le t\le\delta}\left(|\eta_x(t)|^2 + 16M^2\right) + N$ for some $x \in [-1,\delta]$.

Hence $\sup_{-1\le x\le\delta} |\eta_x(x)|^2 \le 16M^2 + \frac{N}{1 - \frac{\delta+1}{8}} < 16M^2 + \frac{8N}{7}$ for every δ with $-1 < \delta < x_0$. Thus $\eta'(x)$ is bounded when $x \in [-1, x_0]$, and $\eta(x) = \alpha + \int^{x} \eta'(t)dt$ is bounded which contradicts to $|\eta(x)| \uparrow \infty$ as $x \to x_0$. Therefore, we can conclude that $\eta(x)$ is bounded in [-1,1] and the solution of the given equation exists.

We have shown that the solutions of (2) always exist for $x \in \Re$ and these solutions are bounded. Since we assume that $\eta(-\infty) = 0$, only two types of solutions, $\eta(x) = 0$ and the solitary wave solutions, can appear for $x < x_-$.

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