SHORT REPORTS

ON ITERATIVE ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper two theorems of existence and uniqueness of the solution of the system of first order iterative ordinary differential equations will be stated and proved. One theorem and one corollary on the existence and uniqueness of the solution of the infinite series of the iterative first order ordinary differential equation will also be given. One example will be illustrated and one problem will be posted.

I. INTRODUCTION

The system of first order iterative ordinary differential equations in the interval [0,a] is of the form

$$(1) \qquad dy/dx = f(x, y(x), y^2(x), ..., y^m(x))$$

with the initial condition

$$(2) \qquad y(0) = c$$

where m is a positive integer greater than 1 and

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ . \\ . \\ . \\ y_n(x) \end{bmatrix}, \qquad dy/dx = \begin{bmatrix} dy_1/dx \\ dy_2/dx \\ . \\ . \\ . \\ dy_n/dx \end{bmatrix}, \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix},$$

$$f(x, y(x), y^2(x), ..., y^m(x)) = \begin{bmatrix} f_1(x, y(x), y^2(x), ..., y^m(x) \\ f_2(x, y(x), y^2(x), ..., y^m(x) \\ . \\ . \\ . \\ f_n(x, y(x), y^2(x), ..., y^m(x) \end{bmatrix}$$

and

$$y^2(x) = \begin{bmatrix} y_1(y_1(x)) \\ y_2(y_2(x)) \\ . \\ . \\ . \\ y_n(y_n(x)) \end{bmatrix}$$

$$y^3(x) = \begin{bmatrix} y_1(y_1{}^2(x)) \\ y_2(y_2{}^2(x)) \\ . \\ . \\ . \\ y_n(y_n{}^2(x)) \end{bmatrix}$$

$$\vdots$$

$$y^m(x) = \begin{bmatrix} y_1(y_1{}^{m-1}(x)) \\ y_2(y_2{}^{m-1}(x)) \\ . \\ . \\ . \\ y_n(y_n{}^{m-1}(x)) \end{bmatrix}$$

and $y_i : [0,a] \to [0,a]$, $y : [0,a] \to R^n$

$f_i : D \to R$, $f : D \to R^n$ and $D = [0,a]^{mn+1}$

for $i = 1, 2, \ldots, n$ and $a, c_1, c_2, \ldots, c_n$ are real numbers.

If f is continuous then the problem (1) - (2) is equivalent to the continuous solution of the integral equation.

$$(3) \qquad y(x) = c + \int_o^x f(t, y(t), y^2(t), \ldots, y^m(t))dt$$

II. THEOREMS ON SYSTEM OF FIRST ORDER ORDINARY ITERATIVE DIFFERENTRAL EQUATIONS.

Let consider the function F where

$$(4) \qquad F(y)(x) = c + \int_o^x f(t, y(t), y^2(t), \ldots, y^m(t))dt$$

and

$$P = \{y \mid y:[0,a] \to [0,a], ||y(x)|| \leq k, ||y(x) - y(\bar{x})|| \leq M|x - \bar{x}|\}$$

and

$$||c|| \leq L \leq K$$

where the norm $||.||$ is the Eucledian norm and K, L, M are in R^+, thus

$$F : P \to P .$$

again let

$$Q = \min \{ m, \frac{(K - L)}{a} \}$$

then we have the following theorem.

Theorem 1. Let f be continuous in D and $||f|| < Q$ then there is at least one solution to the problem (1) - (2).

Proof

$$\begin{aligned} ||F(y)(x)|| &= ||c + \int_o^x f(t, y(t), y^2(t), \ldots, y^m(t))dt|| \\ &\leq ||c|| + \int_o^x ||f(t, y(t), y^2(t), \ldots, y^m(t))||dt \\ &\leq L + aQ \\ &\leq K . \end{aligned}$$

and

$$\begin{aligned} ||F(y)(x) - F(y)(\bar{x})|| &\leq || \int_{\bar{x}}^x f(t, y(t), y^2(t), \ldots, y^m(t))dt|| \\ &\leq \int_{\bar{x}}^x ||f(t, y(t), y^2(t), \ldots, y^m(t)||dt \\ &\leq Q|x - x| . \end{aligned}$$

Then, by Schauder Fixed point Theorem, P has at least one fixed point. That is, there exists at least one function $y = y(x)$ in P such that

$$y(x) = c + \int_0^x f(t, y(t), y^2(t), \ldots, y^m(t))dt .$$

Thus there exists at least one solution $y = y(x)$ to the problem (1) - (2). This ends the proof of theorem 1.

Now let $N < 1$ and $T = \min\{N,Q\}$, then we have the following theorem.

Theorem 2. Let f be continuous and $||f|| \leq T$ then there exists a unique solution to the problem (1) - (2).

Proof. Again

$$\begin{aligned} ||F(y)(x)|| &\leq ||c|| + \int_0^x ||f(x, y(t), y^2(t), \ldots, y^m(x))||dt \\ &\leq L + aQ \\ &\leq K . \end{aligned}$$

And

$$\begin{aligned} ||F(y)(x) - F(y)(\bar{x})|| &\leq \int_{\bar{x}}^x ||f(t, y(t), y^2(t), \ldots, y^m(t))||dt \\ &\leq T\,|x - \bar{x}| \\ &\leq |x - \bar{x}| . \end{aligned}$$

Thus, by the Banach Contraction Principle, F has a unique fixed point. That is, there exists a unique solution $y = y(x)$ to the problem

(1) - (2). This ends the proof of theorem 2.

III. INVESTIGATION ON THE NUMBERS K, L, M, Q AND T

We will consider, in the interval [0,1], the equation

$$(5) \qquad \begin{aligned} y_1'(x) &= y_2(x) \\ y_2'(x) &= \frac{1}{2} - \frac{x}{8} + \frac{x^2}{8} - \frac{x^4}{64} - y_1(x) - y_1^2(x) - \frac{y_2^2(x)}{2} \end{aligned}$$

with the initial conditions

$$(6) \qquad \begin{aligned} y_1(0) &= 0 \\ y_2(0) &= 0 \end{aligned}$$

The solution of the problem (5) - (6) is

$$\begin{aligned} y_1(x) &= \frac{x^2}{4} \\ y_2(x) &= \frac{x}{2} \end{aligned}$$

then

$$y_1^2(x) = \frac{x^4}{64} \quad \text{and} \quad y_2^2(x) = \frac{x}{4}$$

and

$$f(x,y,(x), y_2(x), y_1^2(x), y_1^2(x)) = \begin{bmatrix} \frac{x}{2} \\ \frac{1}{2} - \frac{x^2}{8} - \frac{3x^4}{128} \end{bmatrix}$$

Thus we have

$$||f|| \leq \sqrt{0.5}\ , \ ||y(x) - y(x)|| \leq \sqrt{0.5}\ |x - x| \ \text{ and } \ ||c|| = 0.0\ .$$

Hence

$$K = \sqrt{0.5}\ , L = 0.0\ , M = \sqrt{0.5}\ , Q = \sqrt{0.5}$$

and

$$\sqrt{0.5} \leq N < 1.0 \quad \text{and} \quad T = \sqrt{0.5}\ .$$

IV. INFINITE SERIES OF ITERATIVE ORDINARY DIFFERENTIAL EQUATION

The infinite series of the interative first order ordinary differential equation in the interval [0,1] is of the form

$$(7) \qquad y^1(x) = a_0 + a_1 y(x) + a_2 y^2(x) + a_3 y^3(x) + \ldots$$

with the initial condition

$$(8) \qquad y(0) = c$$

where $c, a_0, a_1, a_2, \ldots$ are real numbers.

We are looking for the solution $y = y(x)$ in the interval [0,1] of the problem (7) - (8) where

$$(9) \qquad y^k(x) \text{ is in } c'[0,1]$$

and

$$(10) \qquad |y^k(x) - y^k(\bar{x})| \leq |x - \bar{x}|$$

for all $x, \bar{x}$ in [0,1] and for all $k = 1, 2, 3, \ldots$.

Thus we have

$$\left|\sum_{k=0}^{\infty} a_k y^k(x)\right| \leq \sum_{k=0}^{\infty} |a_k|\ |y^k(x)|$$

$$\leq \sum_{k=0}^{\infty} |a_k|\ .$$

Thus if $\sum_{k=0}^{\infty} |a_k|$ converges then $\sum_{k=0}^{\infty} a_k\, y^k(x)$ converges uniformly to the continouous function, f(x) and if $\sum_{k=0}^{\infty} |a_k|$ converges to the number K then we have

$$(11) \qquad |f(x)| \leq K$$

and

$$|f(x) - f(x)| \leq \sum_{k=0}^{\infty} |a_k|\, |y^k(X) - y^k(\bar{x})|$$

$$\leq \sum_{k=0}^{\infty} |a_k|\, |x - \bar{x}|$$

thus

$$(12) \quad |f(x) - f(\bar{x})| \leq K|x - \bar{x}| \;.$$

Now let us consider the integral equation

$$(13) \qquad y(x) = c + \int_o^x \sum_{k=0}^{\infty} a_k\, y^k(t)dt \;.$$

If the above conditions are satisfied then there exists a continuous function f(x) such that

$$y(x) = c + \int_o^x f(x)dt \;.$$

Now we have

$$(14) \qquad |c + \int_o^x f(t)dt| \leq 1$$

for all x in [0,1] or

$$(15) \qquad c + K \leq 1$$

then we have the following theorem and corollary.

Theorem 3. If $\sum_{k=0}^{\infty} a_k$ converges absolutely and the conditions (9) - (15) are satisfied then there exists a unique continuous solution y = y(x) to the problem (7) - (8) .

Corollary. If $|a_k| \leq \frac{1}{2^{k+2}}$ and $0 \leq c \leq \frac{1}{2}$ then there exists a unique continuous solution $y = y(x)$ to the problem (7) - (8) .

V. Example

Find the solution in the interval [0,1] of the following differential equation

(16) $$y'(x) = -\frac{3}{2}\left[\frac{y(x)}{2} + \frac{y^2(x)}{2} + \frac{y^3(x)}{8} + \frac{y^4(x)}{8} + \frac{y^5(x)}{32} + \frac{y^6(x)}{32} + \dots\right]$$

with the initial condition

(17) $$y(0) = 1 .$$

We will use the formula (1.17.1) in [1] to find the solution of the given equation.

Let $y_0(x) = 1$

then we get

$$y_1(x) = 1 - \frac{3}{2}\int_o^x \left[\frac{1}{2} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} + \dots\right] dt$$

$$= 1 - \frac{3}{2}\int_o^x \frac{2}{3}\,dt$$

$$= 1 - x$$

and

$$y_2(x) = 1 - \frac{3}{2}\int_o^x \left[\frac{1-x}{2} + \frac{x}{2} + \frac{1-x}{8} + \frac{x}{8} + \dots\right] dt$$

$$= 1 - x$$

and

$$y_3(x) = 1 - x$$

$$y_4(x) = 1 - x$$

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$$y_k(x) = 1 - x$$

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thus the solution of the given equation is

$$y(x) = 1 - x .$$

VI. Problem

One problem that interested us very much is

(18) $$y'(x) = \frac{1}{2} + \frac{y(x)}{4} + \frac{y^2(x)}{8} + \frac{y^3(x)}{16} + \frac{y^4(x)}{32} + \ldots..$$

where x in [0,1] and with the initial condition

(19) $$y(0) = 0 .$$

REFERENCE

1. Maitree Podisuk, Iterative Differential Equations, Ph.D. Dissertation, Institute of Mathematics, Jagiellonian University, Ul. Reymonta 4, 30-059 Krakow, Poland, 1992.