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POWER SERIES OF EISENSTEIN-HURWITZ-CARLITZ TYPE

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ABSTRACT

A class of formal power series generalizing Hurwitz series and series satisfying the Eisenstein condition are considered. This class contains a subclass of power series satisfying linear differential equations with polynomial coefficients not having 0 as one of their singularities. Basic algebraic properties are investigated, especially an Eisenstein-like criterion is established.

INTRODUCTION

Eisensteinian series (or series satisfying the Eisenstein condition) are formal power series of the form $\Sigma\ c_n z^n/I^n$, where c_n are rational integers and I is a fixed positive integer. It is well-known,[1] that algebraic power series with rational coefficients are Eisensteinian; see ref.2 for a generalization. Hurwitz series are formal power series of the form $\Sigma\ c_n z^n/n!$, where c_n are rational integers. They were first introduced by Hurwitz[3] in connection with his work on the coefficients of lemniscate functions, and have since been subject to a number of varied investigations. One obvious yet interesting feature of Hurwitz series is that itself and all its derivatives take rational integral values at the origin. In an attempt to establish an Eisenstein criterion for Hurwitz series, Carlitz[4] was led to *the class* G_s of power series of the form $\Sigma\ c_n z^n/[n,s]$, where c_n are rational integers, s a fixed nonnegative integer, and

$$[n,s] = \frac{n!(n+1)!\dots(n+s)\ !}{1!2!\dots s!}\ ,\ [n,0] = n!\ ,\ [0,s] = 1$$

We now introduce our object of study. A formal power series is said to be an Eisenstein-Hurwitz-Carlitz series of (s,I) type (or is said to *belong to the class E(s;I))* if it is of the form $\Sigma\ c_n z^n/I^n[n,s]$, where c_n are *algebraic integers,* s and I (≥ 1) are nonnegative rational integers. We are interested in this class of series not only because of its pathological nature, i.e. its generalized form over the Eisensteinian and Hurwitz series, but we were originally led to them through our study of power series solutions of certain linear differential equations. Indeed, it is not difficult to verify that if a power series y with algebraic coefficients satisfies a linear differential equation of the form

$$P_K(z)y^{(K)} + \dots + P_1(z)y' + P_0(z) = 0,$$

where $P_i(z)$, $i=0,1,\ldots,K$, are polynomials with algebraic coefficients, and if $P_K(0) \neq 0$, then $y \in E(0;l)$ for some fixed positive integer l; see ref.5, 6 for further related results. It is therefore worth-while to examine basic algebraic properties of this class of power series, particularly, in relation to the Eisenstein criterion, its ancestral birth-place. Of course, some of the results given below can simply be deduced from those in Carlitz[4] by a change of variable, because $f(z) \in E(s;l)$ if and only if $f(lz) \in G_s$, but contains a few errors and the Eisenstein condition deducible from that is somewhat weaker than what we shall give in our second proof of Theorem 3 below. The exposition of this paper is self-contained, except for the two proofs of Theorem 3 where we quote from ref.2 a few lemmas whose proofs are too long to reproduce here.

METHODS.

First, we prove a formula of Zeipel (Lemma 2), which is mentioned without proof in Carlitz.[4]

Lemma 1. Let p and k be nonnegative rational integers. Then

$$d_k := \begin{vmatrix} 1 & \binom{p}{1} & \cdots & \binom{p}{k} \\ 1 & \binom{p+1}{1} & \cdots & \binom{p+1}{k} \\ \vdots & & & \\ 1 & \binom{p+k}{1} & \cdots & \binom{p+k}{k} \end{vmatrix} = 1$$

Proof. Subtracting the second row from the first row, the third from the second,..., the (k+1)th from the kth row, we get

$$d_k = \begin{vmatrix} 0 & -\binom{p}{0} & \cdots & -\binom{p}{k-1} \\ 0 & -\binom{p+1}{0} & \cdots & \binom{p+1}{k-1} \\ \vdots & & & \\ 0 & -\binom{p+k-1}{0} & \cdots & -\binom{p+k-1}{k-1} \\ 1 & \binom{p+k}{1} & \cdots & \binom{p+k}{k} \end{vmatrix}$$

Expanding this last determinant by the first column, we get

$$d_k = d_{k-1}$$

Continuing recursively, we obtain

$$d_k = d_{k-1} = \cdots = d_1 = \begin{vmatrix} 1 & \binom{p}{1} \\ 1 & \binom{p+1}{1} \end{vmatrix} = 1$$

Lemma 2. Let m and k be nonnegative rational integers, $0 \leqslant i \leqslant m$. Then

$$\det\left[\binom{m+r}{i+s}\right]_{r,s=0,\dots,k} = \binom{m}{i}\binom{m+1}{i}\cdots\binom{m+k}{i}/\binom{i}{0}\binom{i+1}{1}\cdots\binom{i+k}{k} = [m,k]/[i,k][m-i,k].$$

Proof. Writing D for the determinant under discussion. Then

$$\binom{i}{0}\binom{i+1}{1}\cdots\binom{i+k}{k}D = \begin{vmatrix} \binom{m}{i}\binom{i}{0} & \binom{m}{i+1}\binom{i+1}{1} & \cdots & \binom{m}{i+k}\binom{i+k}{k} \\ \binom{m+1}{i}\binom{i}{0} & \binom{m+1}{i+1}\binom{i+1}{1} & \cdots & \binom{m+1}{i+k}\binom{i+k}{k} \\ \vdots & & & \\ \binom{m+k}{i}\binom{i}{0} & \binom{m+k}{i+1}\binom{i+1}{1} & \cdots & \binom{m+k}{i+k}\binom{i+k}{k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{m}{i} & \binom{m}{i}\binom{m-i}{1} & \cdots & \binom{m}{i}\binom{m-i}{k} \\ \binom{m+1}{i} & \binom{m+1}{i}\binom{m+1-i}{1} & \cdots & \binom{m+1}{i}\binom{m+1-i}{k} \\ \vdots & & & \\ \binom{m+k}{i} & \binom{m+k}{i}\binom{m+k-i}{1} & \cdots & \binom{m+k}{i}\binom{m+k-i}{k} \end{vmatrix}$$

Factoring a common factor from each row, and making use of Lemma 1, we get

$\binom{i}{0}\binom{i+1}{1}\cdots\binom{i+k}{k}D = \binom{m}{i}\binom{m+1}{i}\cdots\binom{m+k}{i}$, as desired.

Remarks. Since D is a rational integer, then $[m,k]/[i,k][m-i,k]$ is a rational integer.

Two simple observations about the class E(s;I) are in order. For nonnegative integer s and positive integers I, J, we have

$$E(s;I) \subseteq E(s;J) \text{ if } I/J.$$

For positive integer I and nonnegative integers s,t if $s \leq t$, then

$$E(s;I) \subseteq E(t;I).$$

Note also that E(0;1) is the class of (generalized) Hurwitz series, and if we define $[n,-1] = 1$, then $E(-1;I)$ is the class of (generalized) Eisensteinian series with parameter I.

Theorem 1. The class E(s;I) is closed under addition, subtraction and multiplication, when s is a nonnegative integer and I is a positive integer.

Proof. That E(s;I) is closed under addition and subtraction is obvious, so we need only check the multiplication. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n / I^n [n,s] \ , \ g(z) = \sum_{n=0}^{\infty} b_n z^n / I^n [n,s]$$

be elements of E(s;I), and so a_n and b_n are algebraic integers. Putting

$$f(z)g(z) := \sum_{n=0}^{\infty} c_n z^n \ ,$$

then

$$c_n = \sum_{i=0}^{n} \frac{a_i}{I^i[i,s]} \cdot \frac{b_{n-i}}{I^{n-i}[n-i,s]}$$

Therefore,

$$I^n[n,s]c_n = \sum_{i=0}^{n} a_i b_{n-i} \ [n,s]/[i,s][n-i,s] \ .$$

By Zeipel's formula (Lemma 2), each term in the sum on the right hand side is an algebraic integer and hence f(z)g(z) belongs to E(s;I).

The closure property does not quite hold for division in E(s;I), but it is close enough as we shall see next. We need yet another definition.

Definition. A power series $\Sigma\, c_n z^n$ is said to be of **order** r if $c_0 = c_1 = \ldots = c_{r-1} = 0$, but $c_r \neq 0$.

Theorem 2. Suppose f(z) and g(z) are elements of E(s;I) and f(z) is of order r, while g(z) is of order $\geq r$. Then there exists a natural number c such that $cg(z)/f(z) \in E(r+s;Ic)$ and so $cg(cz)/f(cz) \in E(r+s;I)$.

Proof. Since $f(z)\epsilon E(s;I)$ is of order r, then it can be written as

$$f(z) = \sum_{m=0}^{\infty} f_{m+r}\, z^{m+r}/I^{m+r}[m+r,s] = z^r I^{-r}[r,s]^{-1} \sum_{m=0}^{\infty} f_{m+r} z^m [r,s]/I^m[m+r,s],$$

where all f_i $(i=r,r+1,...)$ are algebraic integers, $f_r \neq 0$. Since

$$[r,s]/[m+r,s] = [m,r-1]/[m,r+s],$$

then f(z) can be rewritten as

$$f(z) = z^r I^{-r}[r,s]^{-1} \sum_{m=0}^{\infty} a_m z^m/I^m[m,r+s] \ ,$$

where the a_m are algebraic integers, and $a_0 \neq 0$. Notice also that the series on the right hand side belongs to the class $E(r+s;I)$. In the same manner, we can write

$$g(z) = z^r I^{-r}[r,s]^{-1} \sum_{k=0}^{\infty} d_k z^k/I^k[k,r+s] \ ,$$

where all d_k are algebraic integers. Let

$$g(z)/f(z) = \sum_{j=0}^{\infty} g_j z^j/I^j[j,r+s],$$

where the g_j are now algebraic numbers. Then equating the coefficients, we get

$$\sum_{m=0}^{k} a_m g_{k-m}\ [k,r+s]/[m,r+s][k-m,r+s] = d_k \qquad \text{..........................(1)}$$

For k=0, we see that $a_0 g_0 = d_0$, which is an algebraic integer. For k=1, we have

$$a_0 g_1 + a_1 g_0 = d_1, \text{ an algebraic integer },$$

so that $a_0^2 g_1$ is algebraic integral. Now we proceed by induction. Assume that $a_0 g_0$, $a_0^2 g_1$, ... , $a_0^k g_{k-1}$ are algebraic integers. From (1),

$$a_0 g_k + \sum_{m=1}^{k} a_m g_{k-m}[k,r+s]/[m,r+s][k-m,r+s] = d_k, \text{ an algebraic integer .}$$

Using Zeipel's formula and the induction hypothesis, we see that $a_0^{k+1} g_k$ is an algebraic integer. Replacing a_0 by $c:=|Norm(a_0)|$, which is a natural number, we still get an algebraic integer $c^{k+1}g_k$. Consequently, $cg(z)/f(z)\epsilon E(r+s;Ic)$, as required. The last assertion follows immediately from this.

Corollary 1. If f(z), g(z) are elements of E(s;I), f(z) is of order r, and g(z) is of order at least r, then there exist positive integers c and $C(\geq r)$ such that $cg(z)/f(z)\epsilon E(1;s+C)$.

Proof. By Theorem 2, there exists a natural number c such that

$$cg(z)/f(z) = \sum_{j=0}^{\infty} b_j z^j / I^j c^j [j,s+r] ,$$

where the b_j are algebraic integers. For a positive integer $C \geq r$, note that

$$[j,s+C]/[j,s+r] = \binom{s+C+1}{C-r}\binom{s+C+2}{C-r}\cdots\binom{s+C+j}{C-r}(C-r)!^j$$

Now choose C large enough so that $(C-r)!^j/I^j c^j$ is rational integral for all j, the corollary then follows.

RESULTS

In this section, we prove the following result.

Theorem 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with algebraic coefficients. Suppose f(z) satisfies an algebraic equation of the form

$$P_K(z)f^K(z) + \ldots + P_1(z)f(z) + P_0(z) = 0,$$

where K is a natural number, $P_i(z)$ $(i=0,1,\ldots,K)$ are elements of $E(t;I)$, $P_K(z) \neq 0$, and t, $I(\geq 1)$ are nonnegative rational integers. Then there exist natural numbers J, c and a nonnegative rational integer r such that $cf(z) \in E(r+t;J)$.

Remarks. We give two proofs for this theorem. The first proof resembles the proof of Theorem 5 in Carlitz;[4] this proof is short but it can only provide the existence of $r \geq 1$. The second proof though longer yields $r \geq 0$, which is best possible in the sense that $r=0$ can actually occur as can be seen by taking a special case of Theorem 2.

For the first proof, we need a lemma of Carlitz[4] whose proof is a slight modification, via Theorem 1 above, of Lemma 1 in ref. 2 so we omit it here.

Lemma 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with algebraic coefficients. Suppose f(z) satisfies an algebraic equation

$$P_K(z)f^K(z) + \ldots + P_1(z)f(z) + P_0(z) = 0, \quad P_K(z) \neq 0,$$

where $P_i(z)$, $i=0,1,\ldots,K$, are elements of $E(t;I)$, then there exist nonnegative integer s and positive integer J such that $F(z) := \sum_{n=0}^{\infty} a_{s+n+1} z^n$ satisfies

$$\sum_{k=0}^{K} z^{(s+1)k} B_k(z) F^k(z) = 0,$$

where the $B_k(z)$ are elements of $E(t;J)$, $B_k(z) \neq 0$, $B_1(z)$ not divisible by z^{s+1}, and $B_0(z)$ of order at least $s+1$.

First proof of Theorem 3. By Lemma 3, $F(z) = \sum_{n=0}^{\infty} a_{s+n+1}z^n$ satisfies an algebraic equation of the form

$$\sum_{k=0}^{K} C_k(z)F^k(z) = 0, \qquad \text{..........................(2)}$$

where, for some suitable *natural number* $r \geqslant s+1$ depending only on s,

$$C_o(z) = \sum_{j=s+1}^{r-1} C'_{oj} z^j/J^j[j,t] + \sum_{j=r}^{\infty} C'_{oj} z^j/J^j[j,t]$$

$$= \sum_{j=s+1}^{r-1} c'_{0\,j} z^j/J^j[j,t] + z^r J^{-r}[r,t]^{-1} \sum_{m=0}^{\infty} c'_{0\,,m+r} z^m[r,t]/J^m[r+m,t]$$

$$= \sum_{j=s+1}^{r-1} c'_{0\,j} z^j/J^j[j,t] + z^r J^{-r}[r,t]^{-1} \sum_{m=0}^{\infty} c'_{0\,,m+r} z^m[m,r-1]/J^m[m,r+t]$$

$$= \sum_{j=s+1}^{r-1} c'_{0\,j} z^j/J^j[j,t] + z^r J^{-r}[r,t]^{-1} \sum_{m=0}^{\infty} c_{0,m+r} z^m/J^m[m,r+t]$$

$$C_1(z) = \sum_{j=r}^{\infty} c'_j z^j/J^j[j,t] = z^r J^{-r}[r,t]^{-1} \sum_{m=0}^{\infty} c'_{m+r} z^m[r+m,t]/J^M[r+m,t] c'_r \neq 0,$$

$$= z^r J^{-r}[r,t]^{-1} \sum_{m=0}^{\infty} c'_{m+r} z^m[m,r-1]/J^m[m,r+t]$$

$$= z^r J^{-r}[r,t]^{-1} \sum_{m=0}^{\infty} c_{m+r} z^r/J^m[m,r+t] \,,\ c_r \neq 0,$$

and for $k=2,3,\ldots,K$,

$$C_k(z) = \sum_{j=r+k-1}^{\infty} c'_{k\,j} z^j/J^j[j,t] = z^r J^{-r}[r,t]^{-1} \sum_{m=k-1}^{\infty} c'_{k\,,m+r} z^m[r,t]/J^m[m+r,t]$$

$$= z^r J^{-r}[r,t]^{-1} \sum_{m=k-1}^{\infty} c'_{k,m+r} z^m[m,r-1]/J^m[m,r+t]$$

$$= z^r J^{-r}[r,t]^{-1} \sum_{m=k-1}^{\infty} c_{k,m+r} z^m/J^m[m,r+t] \ ;$$

here c'_{0j}, c'_j, c'_{k_j}, $c_{0,m+r}$, c_{m+r}, $c_{k,m+r}$ are algebraic integers.

Write $F(z) = \sum_{n=0}^{\infty} b_n z^n/[n,r+t]$, where $b_n/[n,r+t] = a_{s+n+1}$.

Equating the coefficients of z^r in (2), we get

$$c_{0,r} + c_r b_0 = 0,$$

and so $c_r b_0$ is an algebraic integer. Equating the coefficients of z^{r+1}, we get

$$\frac{c_{0,r+1}}{J[1,r+t]} + \frac{c_r b_1}{[1,r+t]} + \frac{c_{r+1} b_0}{J[1,r+t]} + \frac{c_{2,r+1} b_0^2}{J[1,r+t]} = 0 ,$$

and so $Jc_r^3 b_1$ is an algebraic integer. Equating the coefficients of z^{r+2}, we get

$$\frac{c_{0,r+2}}{J^2[2,r+t]} + \frac{c_r b_2}{[2,r+t]} + \frac{c_{r+1} b_1}{J[1,r+t][1,r+t]} + \frac{c_{r+2} b_0}{J^2[2,r+t]} + \frac{c_{2,r+1} 2 b_1 b_0}{J[1,r+t][1,r+t]}$$

$$+ \frac{c_{2,r+2} b_0^2}{J^2[2,r+t]} + \frac{c_{3,r+2} b_0^3}{J^2[2,r+t-]} = 0,$$

and so using Zeipel's formula, we see that $J^2 c_r^5 b_2$ is an algebraic integer. Continuing by induction in this manner, we have that $J^n c_r^{2n+1} b_n$ is algebraic integral for each $n \geq 1$. Replacing c_r by $c := |\mathrm{Norm}(c_r)|$, which is a positive integer, we see that $J^n c^{2n+1} b_n := b_n'$ is algebraic integral for all ≥ 1. Thus

$$\begin{aligned} f(z) &= \sum_{n=0}^{s} a_n z^n + \sum_{n=0}^{\infty} a_{n+s+1} z^{n+s+1} \\ &= \sum_{n=0}^{s} a_n z^n + \sum_{n=0}^{\infty} b_n' z^{n+s+1} / J^n c^{2n+1} [n,r+t] \\ &= \sum_{n=0}^{s} a_n z^n + \sum_{n=0}^{\infty} \frac{b_n J^{s+1} c^{2s+1} [n+s+1,r+t]}{[n,r+t]} \cdot \frac{z^{n+s+1}}{J^{n+s+1} c^{2(n+s+1)} [n+s+1,r+t]} \\ &= \sum_{n=0}^{s} a_n z^n + \sum_{n=0}^{\infty} f_{n+s+1} z^{n+s+1} / (Jc^2)^{n+s+1} [n+s+1,r+t], \end{aligned}$$

where the f_{n+s+1} are algebraic integers. Let c^* be a natural number such that $c^* a_n [n,r+t]$ are algebraic integral for $n=0,1,\ldots,s$ and $c^2 | c^*$. Then it is easily checked that $c^* f(z) \in E(r+t; Jc^*)$. Writing c for c^* and J for Jc^*, the theorem is thus proved.

To give the second proof, we need two more lemmas whose proofs are omitted for they follow the same lines as Lemmas 2,3,4 in ref.2 with appropriate modifications via Theorem 1 above; these lemmas are due to Popken[7].

Lemma 4. Let K be a natural number. Among K+2 power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad G_k(z) = \sum_{n=0}^{\infty} g_{kn} z^n \quad (k=0,1,\ldots,K),$$

suppose there exists an algebraic relation

$$\sum_{k=0}^{K} G_k(z) f^k(z) = 0.$$

Let r be a *nonnegative* rational integer so chosen that the series $\sum_{k=0}^{K} kG_k(z)f^{k-1}(z)$ has order r. Then there are K+1 power series

$$H_k(z) = \sum_{n=0}^{\infty} h_{kn}z^n \quad (k=0,1,\ldots,K) \qquad \ldots\ldots(3)$$

with the following properties:

(i) If each $G_k(z)$ is an element of $E(s;I)$, then each $H_k(z)$ is an element of $E(s+2r;I_1)$ for some fixed natural number I_1.

(ii) The following relations hold

$$\sum_{k=0}^{K} H_k(z)f_r^k(z) = 0, \quad \sum_{k=1}^{K} kH_k(0)f^{k-1}{}_r(0) \neq 0, \qquad \ldots\ldots(4)$$

where $f_r(z) = \sum_{n=0}^{\infty} a_{r+n}z^n$.

Lemma 5. Let K be a natural number and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad G_k(z) = \sum_{n=0}^{\infty} g_{kn}z^n \quad (k=0,1,\ldots,K)$$

be power series. Suppose there exists an algebraic relation

$$\sum_{k=0}^{K} G_k(z)f^k(z) = 0.$$

Set

$$T := \sum_{k=1}^{K} kg_{k0}a_0^{k+1}$$

Then

$$Ta_n = - \sum_{(q)} g_{kq_1} \ldots a_{q_k} \quad (n \geqslant 1),$$

where the sum on the right hand side extends over (k+1)-tuples of nonnegative integers $(q) := (q_0, q_1, \ldots, q_k)$ subject to

$$0 \leqslant k \leqslant K, \; 0 \leqslant q_0 \leqslant n, \; 0 \leqslant q_j \leqslant n-1 \quad (j=1,\ldots,k), \; \sum_{j=0}^{k} q_j = n.$$

When k=0, the sum is a single term $g_{0q_0} = g_{0n}$.

Second proof of Theorem 3. Suppose f(z) satisfies $\sum_{k=0}^{K} P_k(z)f^k(z) = 0,$

where each

$$P_k(z) = \sum_{k=0}^{\infty} P_{kn}z^n \ (k=0,1,.....,K)$$

is an element of E(t:I). We can assume without loss of generality that the algebraic equation is of lowest character in the sense that f(z) is not a solution of any algebraic equation of degree smaller than K with coefficients belonging to the class E(t;I). Thus there is a rational integer $r(\geqslant 0)$ for which the series

$$\sum_{k=0}^{K} kP_k(z)f^{k-1}(z)$$

has order r. By Lemma 4, there are power series $H_k(z)$, k=0,1,.....,K, as in (3) such that the algebraic relation in (4) hold, where

$$f_r(z) = \sum_{n=0}^{\infty} a_{r+n}z^n := \sum_{n=0}^{\infty} b_n z^n, \text{ say.}$$

Since each $P_k(z) \in E(t;I)$, then by Lemma 4 also we know that each $H_k(z)$ is an element of $E(t+2r;I_1)$ for some fixed positive integer I_1. Further, by Lemmas 4 and 5, we find that

$$T := \sum_{k=1}^{K} kh_{k0}b^{k-1} = \sum_{k=1}^{K} kH_k\ (0)f_r^{k-1}\ (0) \infty 0, \text{ and}$$

$$Tb_n = -\sum_{(q)} h_{kq_0}\ b_{q1} \ldots b_{qk}\ (n \geqslant 1),$$

where the sum on the right hand side runs over the same range as described in Lemma 5. Since each $H_k(z) \in E(t+2r;I_1)$, then there is a natural number J such that Jb_0^k, $Jh_{k0}b_0^k$, [n,t+2r] $J^n h_{kn} b_0^k$ (k=0,1,...,K; $n \geqslant 1$) all are algebraic integers. We now proceed by induction. For n=1, we have

$$Tb_1 = -\sum_{k=0}^{K} h_{k1}b_0^k$$

and so by the definition of J, we find that $[1,t+2r]JTb_1$ is algebraic integral. For n=2, we have

$$Tb_2 = -\sum_{(q)} h_{kq_0} b_{q_q} \ldots b_{q_k}$$

where the sum extends over integers subject to

$0 \leqslant k \leqslant K$, $0 \leqslant q_0 \leqslant 2$, $0 \leqslant q_j \leqslant 1$ (j=1,2,...,K), $q_0+q_1+...+q_k = 2$.

Using the result of the first case, the definition of J and Zeipel's formula, we see that [2,t+2r] $(JT)^3b_2$ is an algebraic integer.

Now assume $[n,t+2r](JT)^{2n-1}b_n$ is algebraic integral up to n, we wish to show that $[n+1,t+2r](JT)^{2n+1}b_{n+1}$ is also an algebraic integer.

From the recurrence relation

$$Tb_{n+1} = -\sum_{(q)} h_{kq_0}\, b_{q_1} \dots b_{q_k},$$

where the sum extends over $O \leqslant k \leqslant K$, $0 \leqslant q_0 \leqslant n+1$, $0 \leqslant q_j \leqslant n$ $(j=1,2,\dots,k)$, $q_0+q_1+\dots+q_k = n+1$. If $k=0$, then the desired result is trivial. Assume then that $k \neq 0$. Next observe that the highest power of J needed is

$$q_0+(2q_1-1)+\dots+(2q_k-1) \leqslant 2n+1,$$

by the restriction on the sum above. Similarly, the highest power of T needed plus the power on the left hand side of the recurrence relation is

$$(2q_1-1)+\dots+(2q_k-1)+1 = 2(n+1-q_0)-k+1 \leqslant 2n+1,$$

by noting in addition that when $q_0=0$, we have $k \geqslant 2$. Finally, the factor needed from $[.,.]$, using Zeipel's formula, is equal to $[n+1,t+2r]$, as required, i.e. by induction, we have that $[n,t+2r](JT)^{2n-1}b_n$ is algebraic integral for all $n \geqslant 1$. Now if T is not a positive integer, since it is algebraic we can determine a positive integer T' such that T'/T is an algebraic integer. Thus $[n,t+2r](JT')^{2n-1}b_n$ is algebraic integral for each $n \geqslant 1$. The final observation is that since $b_n = a_{n+r}$, then by choosing a new natural number J which depends on the old J, and on $a_0,a_1,\dots,a_r$ we can ensure that a_n has the same property as b_n, i.e. $f(z) \in E(t+2r;J)$, which concludes the proof.

From Theorem 3, we can deduce examples of transcendental power series with respect E(t;I). Theorem 4 is an analogue of Theorem 8 in Carlitz.[4]

Theorem 4. The series $f(z) = \sum_{m=0}^{\infty} z^m/h^{m^2}$, when h is a fixed rational integer $\geqslant 2$, is transcendental relative to E(t;I), with t being a nonnegative rational integer and I a natural number.

Proof. If f(z) is algebraic with respect to E(t;I), then by Theorem 3, there exist natural numbers J, c and a nonnegative rational integer r such that $cf(z) \in E(r+t;J)$ and so we can write

$$ch^{-m^2} = b_m/J^m\,[m,r+t],$$

where the b_m are algebraic integers, and hence rational integers. Let p be a rational prime factor of h. Then the highest power of p dividing $J^m[m,r+t]$ does not exceed

$$\begin{aligned} &km + ([m/p]+[m/p^2]+\dots) + ([(m+1)/p]+[(m+1)/p^2]+\dots) + \dots + \\ &+ ([(m+r+t)/p]+[(m+r+t)/p^2]+\dots) \end{aligned}$$

$$\leqslant km + \frac{m}{p(1-1/p)} + \frac{m+1}{p(1-1/p)} + \dots + \frac{m+r+t}{p(1-1/p)}$$

$$= km + (r+t+1)(2m+r+t)/2(p-1) < k_1 m,$$

where k and k_1 are positive constants independent of m. But evidently, the power of p dividing h^{m^2} is m^2. Consequently, for m sufficiently large h^{m^2} does not divide $J^m[m,r+t]$. This contradiction proves our theorem.

Our last main result is an analogue of the main theorem in Rudin[8] which gives a transcendence test for power series in some subset of E(t+I;I), whose transcendence cannot be deduced from Theorem 3. We follow the same proof as in Rudin.[8]

Theorem 5. Let S be an infinite set of distinct rational primes. Let

$$f(z) = \sum_{m=0}^{\infty} a_m z^m / I^m[m,t+1] \in E(t+1;I),$$

where a_m are rational integers, $I(\geq 1)$ and t are nonnegative rational integers. Assume $a_p \neq 0$ (mod p) for each p in S. For $P_k(z) \in E(t;I)$, k=0,...,K, with rational coefficients, if there is an algebraic relation

$$\sum_{k=0}^{K} P_k(z) f^k(z) = 0, \qquad \text{.........................(5)}$$

then each $P_k(z)$ must vanish identically.

Proof. Assume to the contrary that there is an equation of the form (5), in which not all $P_k(z)$ vanish identically. Applying Lemma 3, it follows that there are nonnegative integers s, $J(\geq 1)$ for which

$$\sum_{k=1}^{K} C_k(z) F^k(z) = C_0(z), \qquad \text{.........................(6)}$$

where $F(z) = \sum_{n=0}^{\infty} a_{n+s+1} z^n / I^{n+s+1}[n+s+1,t+1] \in E(t+1;I)$,

$$C_0(z) = \sum_{j=s+1}^{\infty} c'_{0j} z^j / J^j[j,t] \in E(t;J), \quad c'_{0j} \text{ rational integers},$$

$$C_1(z) = \sum_{j=r}^{\infty} c'_j z^j / J^j[j,t] \in E(t;J), \quad c'_j \text{ rational integers}, \ c'_r \neq 0,$$

$$C_k(z) = \sum_{j=r+k-1}^{\infty} c'_{kj} z^j / J^j[j,t] \in E(t;J), \quad c'_{kj} \text{ rational integers}, \ k=2,3,\ldots,K,$$

and $2s+1 \geq r \geq s+1$. ....(7)

Let L := 1.c.m. (I,J). Then we can rewrite

$$F(z) = \sum_{n=0}^{\infty} f_{n+s+1} z^n / L^{n+s+1} [n+s+1,t+1] \in E(t+1;L),$$

where $f_{n+s+1} = a_{n+s+1}{}^{(L/I)^{n+s+1}}$ are rational integers,

$$C_0(z) = \sum_{j=s+1}^{\infty} c_{0j}z^j/L^j[j,t] \epsilon E(t;L),\ c_{0j} \text{ rational integral,}$$

$$C_1(z) = \sum_{j=r}^{\infty} c_j z^j/L^j[j,t] \epsilon E(t;L),\ c_j \text{ rational integral, } c_r \neq 0,$$

$$C_k(z) = \sum_{j=r+k-1}^{\infty} c_{kj}z^j/L^j[j,t] \epsilon E(t;L),\ c_{kj} \text{ rational integral.}$$

Therefore,

$$C_1(z)F(z) = \sum_{m=r}^{\infty} \frac{z^m}{[m,t]} \sum_{j=r} \frac{c_j[m,t]\ f_{m-j+s+1}}{[j,t][m-j+s+1,t+1]\ L^{m+s+1}} \qquad \text{.........................(8)}$$

By induction, we obtain for $i=2,3,\ldots,K$,

$$C_i(z)F^i(z) = \sum_{m=r+1}^{\infty} \frac{z^m}{[m,t]} \sum \frac{c_{i,u_i}}{L^{m+s+1}} \prod_{j=1}^{i} \frac{[u_{j-1},t]\ f_{u_{j-1}-u_j+s+1}}{[u_j,t][u_{j-1}-u_j+s+1,t+1]}$$

where $u_0=m$, and the second sum is over $r+1 \leqslant u_i \leqslant \ldots \leqslant u_1 \leqslant m$.
Let p be a prime in S satisfying

$$p>|c_r|,\ \ p>r+2+1,\ \ p>(L/I) \qquad \text{.........................(10)}$$

We shall arrive at a contradiction, and hence prove the theorem, by showing that the coefficient of $z^{p+r-s-1}/[p+r-s-1,t]L^{p+r}$ in the left member of (6) is not a rational integer. By (8),(9), this coefficient is a sum of terms consisting of integral multiples of factors of the form

$$[w,t]/[v,t][w-v+s+1,t+1]\ \ (r+1 \leqslant v \leqslant w \leqslant p+r-s-1) \qquad \text{.........................(11)}$$

plus the single term

$$c_r f_p[p+r-s-1,t]/[r,t][p,t+1]. \qquad \text{.........................(12)}$$

But (11) is equal to

$$\frac{1!2!\ldots(t+1)!}{(w-v+s+1)!} \prod_{n=0}^{t} \frac{(w+n)!}{(v+n)!(w-v+s+n+2)!} \qquad \text{.........................(13)}$$

The inequalities (7), (10) and (11) imply
(a) $p \nmid (w-v+s+1)!$
(b) if $p|(v+n)!$, or if $p|(w-v+s+n+2)!$, then $p|(w+n)!$

(c) p^2 does not divide any factorial in (13)

(d) p cannot divide both (v+n)! and (w−v+s+n+2)!.

If follows that none of the terms (11) contains p effectively in the denominator. However, p does appear effectively in the denominator of (12). To see this, we write (12) in the form

$$c_r f_p \frac{1!2!\ldots(t+1)!(p+r-s-1)!(p+r-s)!\ldots(p+r-s+t-1)!}{r!(r+1)!\ldots\ (r+t)!\ p!(p+1)!\ldots(p+t+1)!} \quad \ldots\ldots\ldots\ldots(14)$$

By our choice of p, we have $p \nmid c_r f_p$. As in (c) above, p^2 divides none of the factorials in (14). Hence, it is evident that the numerator of (14) is divisible by p^{t+1} and is not divisible by p^{t+2}, whereas the denominator is divisible by p^{t+2}. This contradiction completes the proof.

Remarks. If we define [m−1] = 1 for nonnegative integer m, then Theorem 5 still holds for t=−1 even when the assumption $a_p \neq 0$ is removed provided f(z) is not a polynomial. This follows readily from the (generalized) Eisenstein theorem proved in.ref. 2.

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บทคัดย่อ

งานวิจัยนี้ศึกษากลุ่มของอนุกรมกำลังที่มีนัยทั่วไปสูงกว่าอนุกรมฮูรวิทซ์และสูงกว่าอนุกรมที่สอดคล้องกับเงื่อนไขของไอเซนสไตน์ กลุ่มของอนุกรมนี้บรรจุอนุกรมกำลังที่เป็นผลเฉลยของสมการเชิงอนุพันธ์เชิงเส้นที่มีสัมประสิทธิ์เป็นพหุนาม และ 0 ไม่เป็นจุดเอกฐาน คุณสมบัติที่ทดสอบคือคุณสมบัติเชิงพีชคณิต โดยเฉพาะอย่างยิ่ง เกณฑ์ประเภทเดียวกับของไอเซนสไตน์ได้รับการตรวจพบว่าเป็นจริง