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A FINITE-PART BOUNDARY INTEGRAL FORMULATION OF AN ELASTIC CRACK PROBLEM

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ABSTRACT

We formulate the antiplane problem concerning a finite elastic body with an arbitrary number of coplanar cracks in its interior in terms of a system of boundary integral equations containing Hadamard finite-part singular integrals. A numerical procedure for solving the integral equations is described. Specific example problems are considered and the numerical results obtained are given.

INTRODUCTION

In recent years, the boundary integral equation method has shown to be a useful and effective numerical tool for analysing linear elastic problems. Nevertheless, the direct application of the method to elastic crack problems encounters some difficulties. To begin with, the crack has to be modelled. For a problem with a certain symmetry (in its geometry and boundary conditions), this may be done by taking only half of the elastic material under consideration with one of the crack faces as part of the boundary. In general, the modelling of the crack is, however, not a trivial task since the opposite crack faces lie on one and the same surface. Cruse[1] has modelled a straight crack as an open-cavity in the shape of an ellipse with a small radius of curvature. This model is unsatisfactory because it gives rise to an almost indeterminate system of linear algebraic equations. Apart from this, there is the difficulty associated with the stress singularities at the crack edges.

An approach to overcome the difficulties discussed above is to modify the fundamental singular solution for the boundary integral equation in such a way that the crack surface is not included in the path of integration. Such a modified fundamental singular solution or Green's function was obtained by Snyder and Cruse[2] and Clements and Haselgrove[3] for a stress-free planar crack in an anisotropic elastic material. Recently, Ang and Clements[4] extended the work of Clements and Haselgrove[3] to include the case where the faces of the planar crack remain wholly in contact throughout the deformation of the material. The Green's function for an arc crack in an isotropic material was given by Ang.[5] The Green's function approach provides highly accurate results, especially in the computation of the stresses near the crack edges. However, the only drawback is that the derivation of the Green's functions requires considerable mathematical effort and it is possible to obtain them explicitly in terms of known elementary functions for a rather limited class of crack problems.

In the present paper, we show how an elastic crack problem can be formulated in terms of boundary integral equations containing Hadamard finite-part singular integrals. For clarity, we will restrict our discussion to the relatively simpler case where the isotropic material contains an arbitrary number of coplanar cracks subject to an antiplane deformation. The formulation presented here contains the crack opening displacement as an unknown function and does not require the modelling of the cracks as open-cavities with small radii of curvature. In addition, due to recent work by Ioakimidis[6] and Kaya and Erdogan,[7] the finite-part boundary integral equations for the problem are easily amenable to numerical treatment. Specific problems are considered and the numerical results obtained are given.

An Elastic Crack Problem

Referred to an Oxyz Cartesian coordinate frame, consider a homogeneous isotropic elastic material R with geometry which is independent of the z coordinate. The interior of the material contains M coplanar cracks in the regions $a_i < x < b_i$, $y=0$, $i=1,2,\ldots,M$, where a_i and b_i are real constants which are such that $a_1 < b_1 < a_2 < b_2 < \ldots < a_M < b_M$ Denote the i-th crack by Γ_i and the exterior boundary of the material by C (Fig. 1).

The material is subject to an antiplane deformation. Specifically, the displacement (u,v,w) is such that u=v=0 and w=w(x,y). The material has elastic behaviour which is governed by the Laplace's equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \tag{2.1}$$

Displacements or tractions independent of z are specified on the exterior boundary C in such a way that the cracks are traction-free. The problem is to find the displacement W satisfying (2.1) and the conditions on the cracks and the boundary C.

Integral Equations

If the displacement W and the traction P are completely known on the boundary $\Omega=C+\Gamma_1+\Gamma_2+\ldots+\Gamma_M$ then a solution of the crack problem is given by (see, e.g. Clements[8])

$$\lambda w(x_0) = \int_\Omega [\Lambda(x,x_0)\ w(x) - P(x)\Phi(x,x_0)]\, dS(x), \tag{3.1}$$

where $\underline{x}=(x,y)$, $x_0=(\xi,\upsilon)$, $\lambda=1$ if $\underline{x}_O$ lies in the interior R and $O < \lambda < 1$ if $\underline{x}_O$ lies on Ω and

$$\Phi(\underline{x},\underline{x}_0) = \frac{1}{4\pi} \log([x-\xi]^2 + [y-\eta]^2),$$

$$\overline{\Lambda(\underline{x},\underline{x}_0)} = \frac{\eta_x(x-\xi) + \eta_y(y-\eta)}{2\pi([x-\xi]^2+[y-\eta]^2)} \tag{3.2}$$

where (η_y,η_y) is the unit normal (outward) vector to R on Ω at the point (x,y). Note that for convenience we have chosen the elastic modulus μ of the material to be one.

Differentiating (3.1) with partially with respect to η, we obtain

$$\lambda w_\eta(\underline{x}_0) = \int_\Omega [\Lambda^* (\underline{x},\underline{x}_o) \ w(x) - P(\underline{x})\Phi^* (\underline{x},\underline{x}_o)] \ dS(\underline{x}), \tag{3.3}$$

where the subscript η denotes partial differentiation with respect to η, and

$$\Phi^* (\underline{x},\underline{x}_o) = \frac{\eta - y}{2\pi([x-\xi]^2 + [y-\eta]^2)}$$

$$\Lambda^* (\underline{x},\underline{x}_0) = \frac{2(y-\eta)(x-\xi)\eta_x - ([x-\xi]^2 - [y-\eta]^2)\eta_y}{2\pi([x-\xi]^2 + [y-\eta]^2)^2} \tag{3.4}$$

If we choose $\underline{x}_0$ to be the (limiting) points in R that approach the top face of each of the cracks and apply the condition P=0 on Γ_i (i=1,2,.....,M), we find that (3.3) becomes

$$\int_C [\Lambda^* (\underline{x},\xi,O^+) \ w(\underline{x}) - P(x)\Phi^* (x,\xi,O^+)] \ dS(\underline{x})$$
$$- \frac{1}{2\pi} \neq\!\!\!\!\int_{a_p}^{b_p} \Delta w(x) \ (x-\xi)^{-2} \ dx - \frac{1}{2\pi} \sum_{i=1,\ i\neq p} \int_{a_i}^{b_i} \Delta w(x) \ (x-\xi)^{-2} \ dx = 0$$

$$\text{for } a_p < \xi < b_p, \ p=1,2,.....,M, \tag{3.5}$$

where $\neq$ denotes that the integral is to be interpreted in the Hadamard finite-part sense and $\Delta w(x) = w(x,0^-) - w(x,0^+)$ for $a_p < x < b_p$, p=1,2,.....,M.

A Numerical Procedure

The integral equations in (3.1) and (3.5) are employed to obtain a numerical procedure for the solution of the crack problem.

The exterior boundary C is first discretised into N straight segments. Assume that the displacement W and the traction P are constant over a given segment. If we take $\underline{x}_O$ to be the midpoint of each of the N segments in turn, after applying the condition P=O on the crack faces, we obtain the approximation

$$\frac{1}{2} w_k = \sum_{n=1}^{N} [W_n \int_{C_n} \Lambda(\underline{x},\underline{p}_k) \ dS(\underline{x}) - P_n \int_{C_n} \Phi(\underline{x},\underline{p}_k) \ dS(\underline{x})]$$
$$- \frac{d_k}{2\pi} \sum_{i=1}^{M} \int_{a_i}^{b_i} \Delta w(x) \ ([x-c_k]^2 + d_k^2)^{-1} \ dx \ \text{for } k=1,2,....,N, \tag{4.1}$$

where C_n denotes the n-th segment of the discretised boundary C, $\underline{P}_k=(c_k,d_k)$ is the midpoint of the k-th segment and w_n and P_n are respectively the constant values of W and P over the n-th segment.

Assuming that the cracks do not intersect with the exterior boundary C, we make the approximation

$$\Delta w(x) \approx \sqrt{1-\left(\frac{x-a_i-L_1}{L_i}\right)^2} \sum_{j=1}^{J} \alpha_J^{(i)} U_{j-1} \left(\frac{x-a_i-L_i}{L_i}\right)$$

$$\text{for } a_i \langle x \langle b_i \;\; i=1,2,.....,M, \tag{4.2}$$

where $2L_i=b_i-a_i$ $U_j(x)$ is the j-th order Chebyshev polynomial of the second kind and $\alpha_j^{(i)}$ are real coefficients yet to be determined.

Substitution of (4.2) into (3.5) yields, after some manipulation (see, e.g. Kaya and Erdogan[7]), we obtain the approximation

$$\sum_{n=1}^{N} [W_n \int_{C_n}^{\Lambda^*} \Lambda^* \; (\underline{x}, L_p s+a_p+L_p, 0^+ \; dS(\underline{x}) - P_n \int_{C_n} \Phi^* \; (\underline{x}, L_p s+a_p+L_p, 0^+) \; dS(\underline{x})]$$
$$+ \sum_{j=1}^{J} \frac{\alpha_j^{(p)}}{L_P} \, j \; U_{j-1}(S) - \frac{1}{2\pi} \sum_{i=1, i\neq p}^{M} \sum_{j=1}^{J} \alpha_j^{(i)} \, L_i \int_{-1}^{1} \frac{\sqrt{1-r^2} \; U_{j-1} \; (r) \; dr}{(L_i r + a_i + L_i - L_p S - a_p - L_p)^2}$$
$$= 0 \text{ for } -1 \langle s \langle 1, \; p=1,2,.....,M. \tag{4.3}$$

Putting (4.2) into (4.1), we obtain

$$\frac{1}{2} w_k = \sum_{n=1}^{N} [w_n \int_{C_n} \Lambda(\underline{x},\underline{p}_k) \; dS(\underline{x}) - P_n \int_{C_n} \Phi \; (\underline{x},\underline{p}_k) \; dS(\underline{x})]$$

$$- \frac{d_k}{2\pi} \sum_{j=1}^{M} \sum_{j=1}^{J} \alpha_j^{(i)} \; L_i \int_{-1}^{1} \frac{\sqrt{1-r^2} \; U_{j-1} \; (r) \; dr}{(L_i r+a_i+L_i-C_k)^2+d_k^2} \text{ for } k=1,2,.....,N. \tag{4.4}$$

If over any segment C^m, either W_m or P_m is specified and if we choose the parameters to be given in turn by

$$s = s_p = \cos[\; (2p-1)\pi/(2J) \;] \text{ for } p=1,2,.....J, \tag{4.5}$$

equations (4.3) and (4.4) constitute a system of (N+JM) linear algebraic equations in (N+JM) unknowns. For example, if W is completely specified on C, the unknowns are P_n (n=1,2,.....,N) and $\alpha_j^{(i)}$ (i=1,2,.....M; j=1,2,.....J). We can now solve for the unknowns from the linear algebraic equations.

In equations (4.3) and (4.4), all the integrals over the segment C_n are computed numerically using the nine-point (extended) trapezoidal rule, except for n=k, for the integrals with the logarithmic integrand, we use

$$2\pi \int_{C_k} \Phi(\underline{x},\underline{p}_k)\, dS(x) = \delta_k(-1+\log(\delta_k/2)), \tag{4.6}$$

where δ_k is the length of the segment C_k. The integrals over the interval (−1,1) in (4.4) are accurately computed using the numerical quadrature (25.4.40) in Abramowitz and Stegun[9].

The stress intensity factors defined by (for i=1,2,.....,M)

$$K_a^{(i)} = \lim_{x \to a_i^-} \sqrt{2(a_i - x)}\ \delta_{yz}(x,O) \quad \text{and} \quad K_b^{(i)} = \lim_{x \to b_i^+} \sqrt{2(x-b_i)}\ \delta_{yz}(x,O), \tag{4.7}$$

where δ_{yz} is the partial derivative of w with respect to y (taking the shear modulus as one), are of fundamental importance in linear fracture mechanics.

From (3.3) and (4.2), the stress intensity factors are approximately given by

$$K_a^{(i)} \approx -\frac{1}{2}\sum_{j=1}^{J} \frac{\alpha_j^{(i)}}{L_i^{1/2}} U_{j-1}(-1) \quad \text{and} \quad K_b^{(i)} \approx -\frac{1}{2}\sum_{j=1}^{J} \frac{\alpha_j^{(i)}}{L_i^{1/2}} U_{j-1}(1). \tag{4.8}$$

Example Problems

Firstly, for a test problem, we take the boundary C to be a square with vertices D(0,2), E(4,2), F(4,−2) and G(0,−2) and choose M=1 (one planar crack) with $a_1=1$ and $L_1=1$ (Fig. 2). Each side of the square is discretised into N_0 straight segments so that $N=4N_0$.

It is easy to verify that the traction which corresponds to the displacement

$$w(x,y) = \mathrm{Re}[i[(x-2+iy)^2-1]^{1/2}], \tag{5.1}$$

where Re denotes the real part of a complex number and $i=(-1)^{1/2}$, vanishes on the crack faces. We use (5.1) to generate displacement on the boundary of the square and then solve for the traction using (4.3) −(4.5). We compare these numerical values of P with those obtained analytically from

$$P = \mathrm{Re}[in_x(x-2+iy)([x-2+iy]^2-1)^{-1/2} - n_y(x-2+iy)([x-2+iy]^2-1)^{-1/2}]. \tag{5.2}$$

At selected points, the comparison is given in Table 1 for N=40 with J=2 and N=120 with J=6. The precentage error in the numerical value of P is computed using

$$(\text{percentage error}) = \left|\frac{\text{numerical value} - \text{exact value}}{\text{exact value}}\right| \times 100\%.$$

TABLE 1. Comparison of numerical values of the traction with the exact ones (test problem).

Point (x,y)	Exact value	N=40, J=2 BIEM	N=40, J=2 % Error	N=120, J=6 BIEM	N=120, J=6 % Error
(0.20,2.00)	−0.9861	−1.0785	9.37	−0.9740	1.23
(0.60,2.00)	−0.9645	−0.9524	1.25	−0.9636	0.09
(1.00,2.00)	−0.9378	−0.9359	0.20	−0.9373	0.05
(1.40.200)	−0.9123	−0.9104	0.21	−0.9120	0.04
(1.80,2.00)	−0.8966	−0.8947	0.21	−0.8962	0.04
(4.00,1.80)	0.0694	0.1219	7.57	0.0739	6.57
(4.00,1.40)	0.0847	0.0892	5.32	0.0856	1.12
(4.00,1.00)	0.0950	0.0981	3.33	0.0954	0.42
(4.00,0.60)	0.0855	0.0877	2.53	0.0858	0.34
(4.00,0.20)	0.0371	0.0380	2.37	0.0372	0.41

TABLE 2. Comparison of the numerical values of the stress intensity factors with the exact ones (test problem).

SIF	N=40,J=2	N=120,J=6	Exact
$K_a^{(1)}$	−0.9998	−1.0000	−1.0000
$K_b^{(1)}$	−0.9998	−1.0000	−1.0000

TABLE 3. Stress intensity factors for various values of α.

SIF	N=60,J=5	N=120,J=10	α
$K_a^{(1)}$	1.3726	1.3801	0.25
$K_b^{(1)}$	1.8038	1.8071	
$K_a^{(1)}$	1.3214	1.3292	0.50
$K_b^{(1)}$	1.5546	1.5577	
$K_a^{(1)}$	1.2726	1.2803	1.00
$K_b^{(1)}$	1.3936	1.3966	
$K_a^{(1)}$	1.2462	1.2539	1.50
$K_b^{(1)}$	1.3308	1.3337	
$K_a^{(1)}$	1.2287	1.2364	2.00
$K_b^{(1)}$	1.2968	1.2997	
$K_a^{(1)}$	1.2171	1.2247	2.50
$K_b^{(1)}$	1.2768	1.2796	
$K_a^{(1)}$	1.2145	1.2220	2.75
$K_b^{(1)}$	1.2719	1.2746	

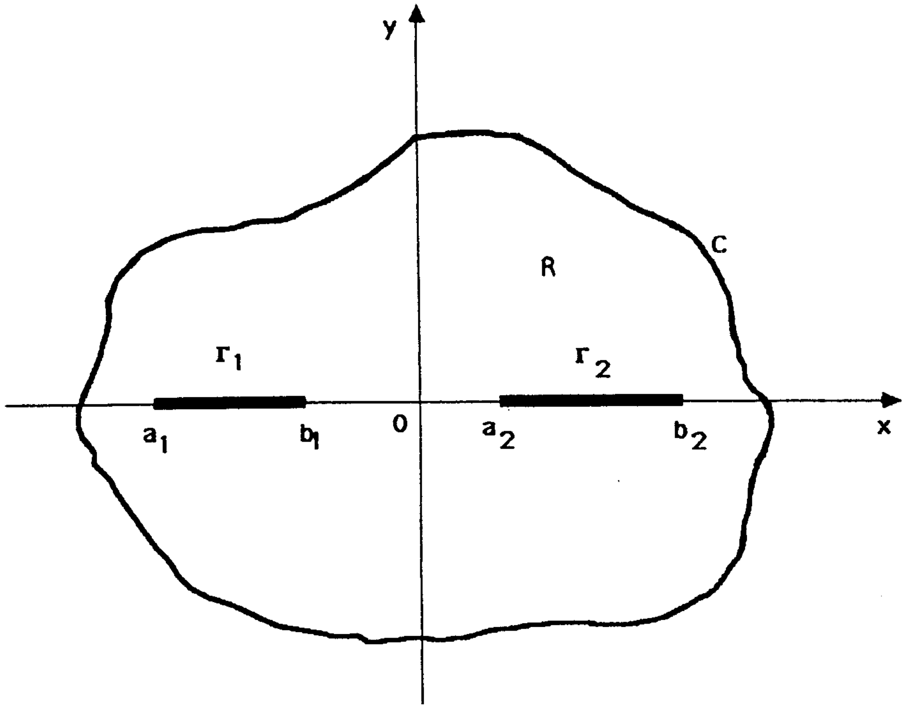

Fig. 1. An elastic material with two coplanar cracks.

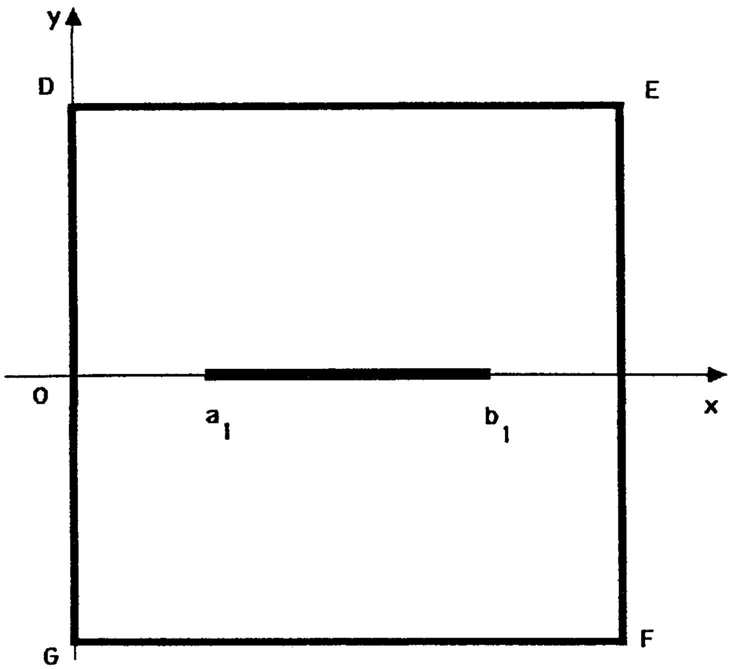

Fig. 2. Sketch for the test problem.

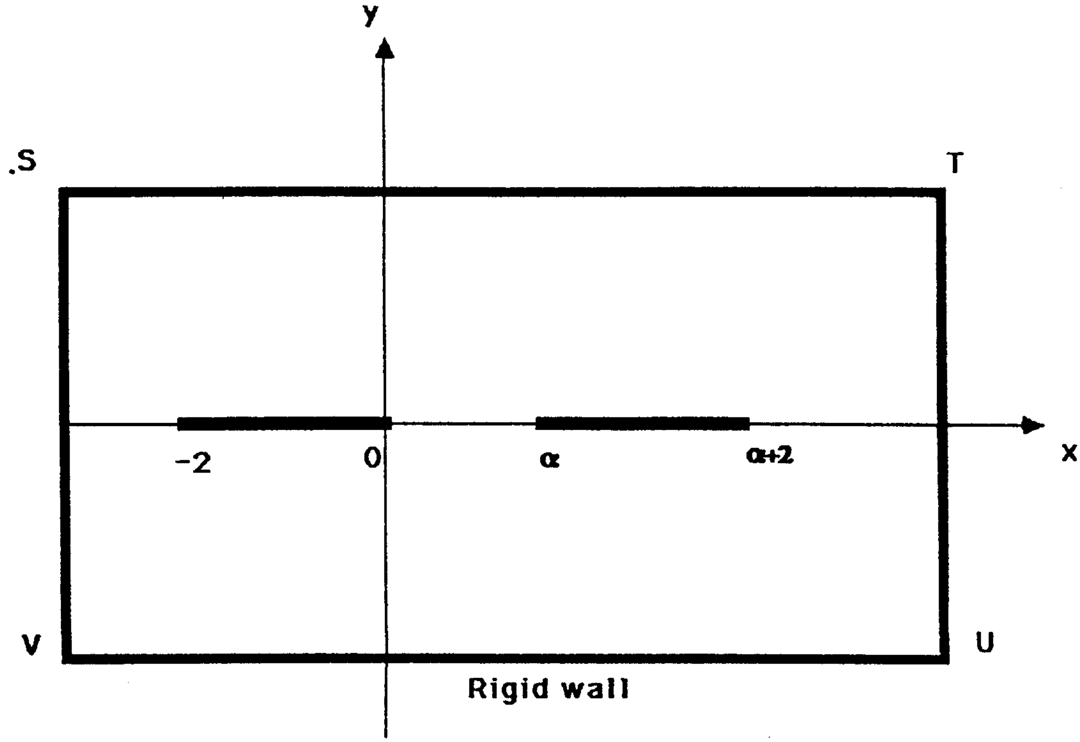

Fig. 3. A rectangular slab with two coplanar cracks.

It is clear from the table that as we increase N and J there is a significant improvement in the numerical values of P.

We also obtain numerical values of the stress intensity factors $K_a^{(1)}$ and $K_b^{(1)}$ using (4.8). The results are given in Table 2. The agreement between the numerical and exact values is excellent.

Consider now the case where the boundary C is a rectangle STUV with S(−3,2), T(5,2), U(5,−2) and V(−3,−2). There are two coplanar cracks in the interior of the rectangular slab. We take $a_1=-2$, $b_1=0$, $a_2=\alpha$ and $b_2=\alpha+2$, where α is a real number between 0 and 3 (Fig. 3). The side UV is attached to a rigid wall so that $w(x,-2)=0$ for $-3<x<5$ and the side ST is subject to shear stress such that $P(x,2)=1$ for $-3<x<5$. The other two remaining sides of the slab are traction-free.

Each of the parallel and vertical sides of the slab is respectively discretised into $2N_1$ and N_1 segments so that $N=6N_1$. After solving (4.3)–(4.5), we compute the stress intensity factors using (4.8). In particular, we are interested in examining how the stress intensity factors $K_a^{(1)}$ and $K_b^{(1)}$ (at the tips $x=-2$ and $x=0$ respectively) are affected by the distance α between the inner tips of the two cracks. In Table 3, the values of these stress intensity factors for various values of α are obtained using N=60 with J=5 and N=120 with J=10. It is clear from the table that increasing α has the effect of decreasing the stress intensity factors $K_a^{(1)}$ and $K_b^{(1)}$

CONCLUSION

We have illustrated how an elastic problem concerning an arbitrary number of coplanar cracks in a finite solid subject to an antiplane deformation can be formulated in terms of a system of finite-part boundary integral equations. A simple procedure for the numerical solution of the integral equations was described. Numerical results were obtained for some specific example problems using the procedure. For one of the problems where analytic solution is known, the numerical results obtained compared favourably with the exact ones. It is possible to extend the formulation presented here to plane problems involving coplanar cracks in general anisotropic materials or to include the case where the planar cracks are arbitrarily-oriented with respect to one another.

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บทคัดย่อ

ได้ศึกษาปัญหาของ coplanar cracks ใน finite elastic body โดยใช้ boundary integral equation ซึ่งมี Hadamard finite-part singular integrals